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Detection Performance of Generalized Likelihood Ratio Processors for Random Signals of Unknown Location, Structure, Extent, and Strength

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PREFACE

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A signal (if present) is located somewhere in a band of frequencies characterized by a total of N search bins. The signal occupies an arbitrary set of M of these bins, where not only is M unknown, but also, the locations of the particular M occupied bins are unknown. Also, the signal strength is unknown.

The generalized likelihood ratio (GLR) method furnishes an approach for estimating all the unknown parameters and for testing for presence or absence of the signal in an observation of the outputs of the N search bins. However, there is nothing guaranteed optimum about the GLR approach. Also, the optimum (likelihood ratio) processor cannot be constructed or realized, due to all the unknowns and the voluminous amount of searching required for this scenario. These deleterious conditions force adoption of some suboptimum processing techniques, guided by the GLR and likelihood ratio results.

14. SUBJECT TERMS
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13. ABSTRACT (continued)

An ad hoc processing technique, called the sum-of-M-largest processor, has been investigated by a combination of analytic means and simulation, in terms of its false alarm and detection probabilities, as functions of N, $\underline{M}$, M, and the average signal-to-noise ratio per bin, $\underline{S}$. This class of processors presumes no knowledge of $\underline{M}$, and has no knowledge of location, structure, or strength information of any sort. The degradation of this technique, due to mismatch between M and $\underline{M}$, has been accurately evaluated quantitatively, without resorting to approximations such as the central limit theorem, which has questionable accuracy for small false alarm probabilities.

Some new results for the characteristic function of a weighted sum of ordered data are derived and used for false alarm probability calculations. Extensions to the characteristic function of the sum of distorted ordered data, as well as to some joint characteristic functions, have also been accomplished.

14. SUBJECT TERMS (continued)

Unknown Strength
Maximum Likelihood
False Alarm Probability
Detection Probability
Order Statistics
Characteristic Function
Simulation

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LIST OF SYMBOLS

N	total number of search bins
$\underline{M}$	actual number of bins occupied by signal
$\underline{L}$	actual set of locations of the occupied bins
$\underline{S}$	actual average signal power per bin
H_0	hypothesis 0, noise-only present
H_1	hypothesis 1, signal and noise present
M	hypothesized number of bins occupied by signal
L	hypothesized occupied set of bins of size M
$\underline{S}_n$	actual average signal power in n-th occupied bin
S_n	hypothesized average signal power in n-th occupied bin
bold	random variable
$\mathbf{x}_n$	n-th observation, data in n-th bin
p_0	probability density function under H_0
p_1	probability density function under H_1
q_0	probability density function of n-th bin under H_0, (1)
q_1	probability density function of n-th bin under H_1, (2)
$\underline{a}_n$	auxiliary parameter, (3)
LR	likelihood ratio, (6)
$\underline{w}_n$	n-th weight, (7)
v	fixed threshold, (8)
a_n	auxiliary parameter, (11)
S	hypothesized common average signal power, (13)
a	auxiliary parameter, (14)
GLR	generalized likelihood ratio, (18)
g	nonlinear transformation, (19)

x'_n	ordered data, (20)
M_1, M_2	bounds on range of M, (22)
a_n	maximum likelihood estimate, (23)
z	sum of M largest random variables, (28)
$f(\xi)$	characteristic function of z under H_0, (29)
$f_1(\xi)$	characteristic function of z under H_0 for M = 1, (30)
$E_z(v)$	exceedance distribution of z under H_1 for M = 1, (31)
$\underline{a}$	auxiliary parameter $(1 + \underline{S})^{-1}$, (31)
overbar	ensemble average, (32)
P_f	false alarm probability, (33)
P_d	detection probability
$\underline{S}$(dB)	required signal power per bin, in dB
x_o	breakpoint, (36)
$g_o(x)$	nonlinearity with breakpoint, (36)
MGLR	modified generalized likelihood ratio

DETECTION PERFORMANCE OF GENERALIZED LIKELIHOOD RATIO PROCESSORS FOR RANDOM SIGNALS OF UNKNOWN LOCATION, STRUCTURE, EXTENT, AND STRENGTH

INTRODUCTION

Reliable detection of weak signals in noise is aggravated when the signal has little or no structure that can be utilized in processing the received waveform. Yet, this is a problem frequently encountered in practical applications and which must be addressed quantitatively in order that attainable performance levels can be established and realized. If significant gains relative to simple energy detection are possible, this fact must be known; also, alternative improved processing techniques must be discovered. Furthermore, the robustness of the alternative techniques to lack of knowledge of the detailed signal characteristics is a critical issue that must be addressed and quantified.

The problem we consider here is couched in the frequency domain, where a known search region of N disjoint bins contain noise which is uniformly distributed over that entire frequency region. In addition, either a signal is present in $\underline{M}$ of those frequency bins, or the signal is absent from all bins. The processing problem is to maximize the detection probability when signal is present, while keeping the false alarm probability fixed at some desirable specified low level.

To complicate the situation, the number of occupied signal bins $\underline{M}$ is unknown; that is, the extent of the signal coverage (total bandwidth) is not known apriori. Furthermore, the $\underline{M}$ occupied signal bins need not be adjacent in frequency or have any discernible pattern in frequency space; that is, the signal spectrum has no usable structure (such as harmonic lines or contiguous bins) that might aid in signal processing and detectability.

Additionally, the actual locations $\underline{L}$ of the particular $\underline{M}$ occupied signal bins (when signal is present) are unknown, except that they must occur somewhere in the total search space of N bins. It is assumed that all of the possible occupancy patterns for the set $\underline{L}$ of $\underline{M}$ occupied bins are allowed.

Finally, the actual average signal powers per bin (presumed equal to a common value $\underline{S}$, for the most part here), are not known; lack of this signal strength information (as well as no knowledge of $\underline{M}$ or $\underline{L}$) precludes realization of any optimum processing technique, which would necessarily rely on and use that information.

The absence of knowledge of these important signal parameters ($\underline{M}$, $\underline{L}$, $\underline{S}$, structure) causes us to adopt maximum likelihood estimation procedures and their attendant generalized likelihood ratio processors. Depending on the particular starting points of the analyses, namely the initial assumptions about the signal parameters, different forms of processors result. This leads to several classes of processors which must be analyzed, either analytically or by simulation. Then, comparisons of these processors are possible and required, thereby enabling establishment of a baseline performance level in this rather deleterious environment. Some recent work along this line is available in [1], where a modified generalized likelihood ratio processor was quantitatively evaluated in terms of its receiver operating characteristics. Knowledge of those procedures and results is presumed of the reader here.

The generalized likelihood ratio processor is not necessarily an optimum procedure for signal detection. Rather, it is an ad hoc procedure frequently adopted for convenience, rationality, and for the fact that it generally yields reasonable processing forms and performance capabilities. However, it must be noted and cautioned there are cases where the generalized likelihood ratio test can actually yield poor performance [2; page 96].

Although the present search and detection problem has been couched in the frequency domain, this is done solely for ease of discussion. The analyses and results actually apply to any search domain, such as time, distance, angle, or combinations of these variables. For example, a typical application could require a search in a combined time, frequency space, where the total search region of N bins would be composed of a rectangular region of size $N = N_t N_f$. However, it will be necessary to investigate if the fundamental assumptions utilized in this study, such as lack of structure, apply in the particular domain(s) of interest to the user.

This technical report is the second of a series of four NUWC technical reports by this author, covering the topics:

(a) modified generalized likelihood ratio processors,

(b) generalized likelihood ratio processors,

(c) power-law processors, and

(d) optimum processing,

respectively. Topic (a) was completed in [1], resulting in a substantial compilation of receiver operating characteristics for the particular modificatior considered there. Topic (b) will be addressed in this report. The overall goal of the extended investigation is to determine classes of processors which perform at or near optimum levels of performance, and which can be easily realized and analyzed, even in these situations of scant knowledge about the detailed signal characteristics.

PROBLEM DEFINITION

The search space consists of N (frequency) bins, each containing independent identically-distributed noises of unit power. This is presumed to be accomplished by an earlier normalization procedure. The number N is under our control and is always a known quantity. When signal is absent, hypothesis H_0, the probability density function of each of the bin outputs is completely known.

When signal is present, hypothesis H_1, the quantity $\underline{M}$ is the actual number of bins occupied by signal. When $\underline{M}$ is unknown, we will hypothesize that M bins are occupied by signal. (If $\underline{M}$ is presumed known, we can take M equal to that presumed known value if we please.)

The quantity $\underline{L}$ is the actual set of bins occupied by signal, when signal is present; for example, if $\underline{M} = 4$, then we might have the set $\underline{L} = \{2,3,7,29\}$. When $\underline{L}$ is unknown, we hypothesize that L is the occupied set of bins, for the previously hypothesized value of M. Thus, the size of set L is equal to M. (If $\underline{L}$ is presumed known, we can take L equal to that presumed known set if we please.)

The quantities $\{\underline{S}_n\}$ are the actual average signal powers per bin in occupied set $\underline{L}$, when signal is present. When these signal powers $\{\underline{S}_n\}$ are unknown, we hypothesize signal powers $\{S_n$ or $S\}$ for hypothesized size M and set L, depending on whether we presume these powers are all different or all equal, respectively. (These average signal powers $\{\underline{S}_n\}$ will usually be unknown in practical applications.)

For these hypothesized quantities M, L, and $\{S_n$ or $S\}$ of the unknown parameters, we can determine the likelihood ratio for a given (random) observation $\{x_n\}$. Then, since the probability density function p_0 of the observation $\{x_n\}$ under hypothesis H_0 is completely known, we can maximize this likelihood ratio (instead of maximizing density p_1 of the observation $\{x_n\}$ under hypothesis H_1) by variation of M, L, and $\{S_n$ or $S\}$, over all their <u>allowed</u> values, thereby obtaining maximum likelihood estimates **M**, **L**, and $\{S_n$ or $S\}$ of the unknown parameters. These estimates are random variables, because they depend on the particular observation $\{x_n\}$. Comparison of this maximum likelihood ratio value, called the generalized likelihood ratio, with a fixed threshold constitutes the generalized likelihood ratio test. Simplifications of this test are frequently possible.

It was mentioned above that when the number of bins $\underline{M}$ occupied by signal is <u>presumed</u> known, we could set M equal to this presumed value, thereby eliminating the search on this parameter. However, a problem with this approach is that, in practical applications, the <u>actual</u> number of occupied signal bins, $\underline{M}$, may be different from the number M presumed during the derivation of the processor. This mismatch between the presumed and actual numbers can lead to a degradation in performance. Quantitative evaluation of this degradation is one of the main topics of this study.

There are three cases that can obtain relative to the available knowledge about the value of $\underline{M}$, the actual number of occupied signal bins. In the first case, $\underline{M}$ could be known exactly; that is, the total signal frequency extent is known exactly, although the precise bin locations and structure are not. This might arise in trying to intercept a frequency-dodging diversity-combining communication message.

In the second case, $\underline{M}$ might be completely unknown; that is, the signal frequency extent could be anything, from a very narrow band (tonals) up to a broad band of frequencies. This situation could occur when there is no apriori knowledge about the signal to be detected. It could also occur in the initial stages of searching for a general signal of unknown center frequency and extent.

Finally, in the third case, size $\underline{M}$ might be partially known. Thus, the signal frequency extent may be known within fairly broad limits, say, for example, from 50 Hertz to 200 Hertz, within a total search band of 1000 Hertz. This situation could obtain when partial information is available about the signal of interest. The three different cases will naturally lead to different processors, each of which makes use of the information available to it.

DERIVATION OF GENERALIZED LIKELIHOOD RATIO TESTS

PROBABILITY DENSITY FUNCTIONS FOR KNOWN SIGNAL PARAMETER VALUES

We now specify the detailed character of the probability density functions p_0 and p_1, introduced above, under hypotheses H_0 and H_1, respectively. In both hypotheses, the bin outputs or observations $\{x_n\}$ are taken as the squared envelopes of outputs of (disjoint) narrowband filters subject to a Gaussian input random process; alternatively, the observations are the magnitude-squared outputs of a fast Fourier transform subject to a Gaussian input process. It is assumed that these outputs $\{x_n\}$ are statistically independent of each other, which is consistent with a frequency-disjoint requirement.

Since the bin output noise has been normalized at unit level, the probability density function of the n-th observation x_n is, under hypothesis H_0, an exponential of the form

$$q_0(u_n) = \exp(-u_n) \quad \text{for } u_n > 0 \ , \quad 1 \le n \le N \ . \tag{1}$$

When signal is present, with signal power $\underline{S}_n$ in the n-th bin, the density of x_n is changed in this signal-present hypothesis H_1, to

$$q_1(u_n) = \underline{a}_n \exp(-\underline{a}_n \, u_n) \quad \text{for } u_n > 0 \ , \quad 1 \le n \le N \ , \tag{2}$$

where we have defined the parameter

$$\underline{a}_n = \frac{1}{1 + \underline{S}_n} \le 1 \quad \text{for } 1 \le n \le N \ . \tag{3}$$

Observe that actual signal power $\underline{S}_n$ can also be interpreted as the actual signal-to-noise power ratio per bin, since the noise power per bin has been normalized at unity.

The probability density function governing the complete observation $\{x_n\}$ under H_0 follows from (1) and the statistical independence as

$$p_0(u_1,\ldots,u_N) = \prod_{n=1}^{N}\{\exp(-u_n)\} \ . \tag{4}$$

On the other hand, under H_1, the pertinent density is, from (2),

$$p_1(u_1,\ldots,u_N) = \prod_{n\varepsilon\underline{L}}\{\underline{a}_n \exp(-\underline{a}_n\ u_n)\} \prod_{n\not\varepsilon\underline{L}}\{\exp(-u_n)\} \ , \tag{5}$$

where $\underline{L}$ is the actual occupied set of signal bins.

If $\underline{L}$ and $\{\underline{S}_n\}$ were all known, the likelihood ratio for observation $\{x_n\}$ would be given by random variable

$$LR \equiv \frac{p_1(x_1,\ldots,x_N)}{p_0(x_1,\ldots,x_N)} = \prod_{n\varepsilon\underline{L}}\Big(\underline{a}_n \exp([1-\underline{a}_n]\ x_n)\Big) =$$

$$= \prod_{n\varepsilon\underline{L}}\{\underline{a}_n\} \exp\Big(\sum_{n\varepsilon\underline{L}} \underline{w}_n\ x_n\Big) \ , \tag{6}$$

where we have defined weights

$$\underline{w}_n = 1 - \underline{a}_n = \frac{\underline{S}_n}{1 + \underline{S}_n} \quad \text{for } n\varepsilon\underline{L} \ . \tag{7}$$

Therefore, the likelihood ratio test in this ideal case is given by the weighted linear sum comparison with fixed threshold v:

$$\sum_{n\varepsilon\underline{L}} \underline{w}_n\ x_n \ \substack{>\\<}\ v \ . \tag{8}$$

PROBABILITY DENSITY FUNCTIONS FOR UNKNOWN SIGNAL PARAMETER VALUES

Unfortunately, the optimum test in (8) cannot be realized in practice, because the occupied set $\underline{L}$ will not be known and the signal strengths $\{\underline{S}_n\}$ in the occupied bins will not be known. To circumvent these problems, we will now consider employing maximum likelihood estimates for the unknown parameters.

When signal is present, suppose we hypothesize that:

$$\begin{aligned}&(1)\ M \text{ bins are occupied by signal,}\\&(2)\ L \text{ is the specific set of } M \text{ occupied bins, and}\\&(3)\ \{S_n\} \text{ are the signal powers per bin for } n \varepsilon L.\end{aligned} \tag{9}$$

Then, by reference to (5), the probability density function governing the observations under H_1 is

$$p_1(u_1,\ldots,u_N) = \prod_{n\varepsilon L}\{a_n \exp(-a_n u_n)\} \prod_{n\not\varepsilon L}\{\exp(-u_n)\} , \tag{10}$$

where we defined parameters

$$a_n = \frac{1}{1 + S_n} \leq 1 \quad \text{for } 1 \leq n \leq N . \tag{11}$$

On the other hand, under H_0, we simply set all signal powers $\{S_n\}$ to zero, obtaining probability density function (4) again. Since this latter function, p_0, is independent of all the parameters hypothesized above in (9), we can obtain the maximum likelihood estimates by maximizing the likelihood ratio instead of maximizing probability density function p_1. The likelihood ratio, for given random observation $\{x_n\}$, parallels (6) and (7), but using (10) now, namely

$$LR \equiv \frac{p_1(x_1,\ldots,x_N)}{p_0(x_1,\ldots,x_N)} = \prod_{n\varepsilon L}\left(a_n \exp([1-a_n]\ x_n)\right) . \tag{12}$$

At this point, in order to make further progress, we have to consider two different possibilities for the signal strengths; these are:

(1) all powers equal: $S_n = S$ for $n\varepsilon L$; or

(2) all powers different: S_n arbitrary for $n\varepsilon L$. (13)

We will derive the generalized likelihood ratios for both situations, and then extract the corresponding tests for two different cases of knowledge about $\underline{M}$, the actual number of bins occupied by signal.

The problem we are addressing here is a noise in noise problem; that is, the filter bank input is a white Gaussian process under H_0, while it is a colored Gaussian process under H_1. However, the coloring is not known, nor is the extent or strength of the coloring known. This lack of information leads to complications in signal processing and in the subsequent determination of the detection capability.

ALL SIGNAL POWERS EQUAL

Here, we presume that all the signal powers are equal in the occupied bins, and we set $S_n = S \geq 0$ for all n in hypothesized set L. Then, define parameter

$$a = \frac{1}{1 + S} \leq 1 . \tag{14}$$

The likelihood ratio in (12) reduces to

$$LR = a^M \exp\left((1 - a) \sum_{n \in L} x_n\right) , \tag{15}$$

where we used the fact that set L is of size M. An important observation to make immediately from (15) is that the data $\{x_n\}$ will be subject to addition of linear quantities in these observations, regardless of how we choose M and L, at least in this case of presumed equal signal powers.

The value of parameter a that maximizes LR in (15) is random variable

$$a = \min\left(1, M / \sum_{n \in L} x_n\right) , \tag{16}$$

where we satisfied constraint (14). The corresponding signal power estimate is, from (14),

$$S = \max\left(0, \frac{1}{M} \sum_{n \in L} (x_n - 1)\right) . \tag{17}$$

Substitution of estimate (16) in (15) yields the generalized likelihood ratio

$$\text{GLR} = \exp\left[M \; g\left(\frac{1}{M} \sum_{n \varepsilon L} x_n\right)\right] , \tag{18}$$

where monotonically increasing function g is defined by

$$g(x) \equiv \begin{Bmatrix} x - 1 - \ln(x) & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{Bmatrix} . \tag{19}$$

Now, we must maximize the argument of the monotonically increasing function exp in (18) by choice of set L and size M.

If the actual set size $\underline{M}$ is known, then we would take hypothesized value M equal to known value $\underline{M}$; reference to (18) and (19) reveals that we could then concentrate on maximizing the sum in (18), where set L is now restricted to be of size $\underline{M}$. But, set L should obviously then be taken to correspond to the $\underline{M}$ largest members of observation $\{x_n\}$.

At this point, it is expedient to consider ordering the measured data $\{x_n\}$. Specifically, order the given data $\{x_n\}$ from largest to smallest according to

$$x_1' \geq x_2' \geq \cdots \geq x_N' . \tag{20}$$

Thus, x_1' is the largest element of $\{x_n\}$, while x_N' is the smallest element of $\{x_n\}$.

We can now easily achieve the desired maximum of the sum in (18) in terms of the ordered random variables $\{x_n'\}$; namely, the generalized likelihood ratio test takes the form

$$\sum_{n=1}^{\underline{M}} x_n' \gtrless v . \tag{21}$$

That is, we must order the data, add the first $\underline{M}$ ordered variables, and compare with a fixed threshold v. This is the generalized likelihood ratio test for equal signal powers and known occupancy size $\underline{M}$.

This test makes a great deal of sense. The only way that a signal displays its presence in the observed data $\{x_n\}$ is through an increase in the means in $\underline{M}$ bins of unknown location; see (1) and (2), where the mean increases from 1 to $1+\underline{S}_n$ in the n-th bin, when signal is present. Test (21) says to consider the $\underline{M}$ largest data values and see if their sample mean is sufficiently large to declare that a signal is present.

On the other hand, if the actual set size $\underline{M}$ is not known, the generalized likelihood ratio in (18) dictates the test

$$\max_{M_1 \le M \le M_2} \left(M \; g\left(\frac{1}{M} \max_{L} \sum_{n \varepsilon L} x_n \right) \right) = \max_{M_1 \le M \le M_2} \left(M \; g\left(\frac{1}{M} \sum_{n=1}^{M} x_n' \right) \right) \gtrless v \; , \qquad (22)$$

where the size of set L is M, and it is presumed that set size M is known to be within a range of values, $[M_1, M_2]$, which must be searched. Again, the ordered data must be linearly summed, then averaged, but now subjected to monotonic transformation g in (19). Although there is no obvious physical interpretation of test (22), it is easily realized once the given data $\{x_n\}$ has been ordered; the search on M itself is not too time consuming. Operation (22) is the generalized likelihood ratio test for equal signal powers and unknown occupancy size $\underline{M}$.

ALL SIGNAL POWERS ARBITRARY

Here, we presume that hypothesized signal power $S_n \geq 0$ for all n in hypothesized set L of size M. The likelihood ratio was given in (12) for this case, where parameter a_n was given by (11). The value of a_n that maximizes the n-th term in (12) is random variable

$$a_n = \min\{1, 1/x_n\} \quad \text{for } n \varepsilon L \; . \tag{23}$$

The corresponding maximum of (12) is then the generalized likelihood ratio

$$\text{GLR} = \prod_{n \varepsilon L} \{\exp[g(x_n)]\} = \exp\left(\sum_{n \varepsilon L} g(x_n)\right) \; , \tag{24}$$

where function g was defined in (19). It is important to observe here, in the case of arbitrary signal powers, that the given data $\{x_n\}$ is always transformed according to nonlinearity g, prior to being summed over hypothesized set L; this is in contrast to the linear sum (15) for equal signal powers.

Now, we must maximize the argument of the exp function in (24) by choice of set L and its size M. If the actual set size $\underline{M}$ is known, then we would take M equal to $\underline{M}$; then, using the monotonicity of function g, set L in (24) should obviously be taken to correspond to the $\underline{M}$ largest members of data $\{x_n\}$. This leads to the test

$$\sum_{n=1}^{\underline{M}} g(x_n') \gtrless v \; . \tag{25}$$

This is the generalized likelihood ratio test for arbitrary signal powers and known occupancy size $\underline{M}$.

On the other hand, if the actual set size $\underline{M}$ is unknown, the generalized likelihood ratio in (24) dictates the test

$$\sum_{n=1}^{M_2} g(x_n') \gtrless v \, , \tag{26}$$

where it is presumed that there is some upper bound, M_2, on the range of values allowed for M. Processor (26) is the generalized likelihood ratio test for arbitrary signal powers and unknown occupancy size $\underline{M}$.

Both tests, (25) and (26), subject the ordered data $\{x_n'\}$ to a small-signal suppression effect, which is inherent in function g, prior to summation. That is, from (19), $g(x) \sim \frac{1}{2}(x - 1)^2$ for $x \sim 1$. However, if $\underline{M}$ or M_2 is a small fraction of the total search size N, the larger data values in the ordered set will dominate these tests. And since function g in (19) is nearly linear for larger arguments, the sums in (25) and (26) will be essentially linear sums in this situation of small $\underline{M}/N$ or M_2/N, respectively.

SUMMARY OF ANALYTICAL RESULTS FOR ORDERED DATA

The generalized likelihood ratio test for equal signal powers and known occupancy size $\underline{M}$ was given in (21) in the form

$$\sum_{n=1}^{\underline{M}} x_n' \gtrless v , \tag{27}$$

where $\{x_n'\}$ is the ordered version of given data $\{x_n\}$. However, when $\underline{M}$ is unknown, this test cannot be realized. Nevertheless, it does suggest a closely related alternative; namely, for hypothesized size M, consider the sum of the M largest random variables, where M is a best guess or mid-range value of $\underline{M}$. That is, consider the decision variable z given by

$$z \equiv \sum_{n=1}^{M} x_n' \gtrless v . \tag{28}$$

The performance of this sum-of-M-largest processor will depend upon both the hypothesized size M as well the actual size $\underline{M}$ of the occupied signal bins, in addition to the actual signal power per bin, $\underline{S}$. Evaluation of the receiver operating characteristics of this processor will occupy much of the remaining effort here.

Although the original data $\{x_n\}$ is independent identically-distributed exponential random variables under H_0, the ordered data $\{x_n'\}$ is <u>not</u> independent, <u>not</u> identically distributed, and <u>not</u> exponential or Gaussian. Thus, (28) constitutes the classical problem of finding the statistics of the output of a digital filter subject to a non-Gaussian input process with dependent data values.

The most important analytical problem of interest here is in finding the first-order statistics of random variable z defined in (28). This problem is solved in appendix A, in particular in (A-11), where the characteristic function of z is derived exactly in closed form for hypothesis H_0, namely independent identically-distributed exponential random variables for original input $\{x_n\}$. At the same time, with the framework already established, a number of extensions have been analyzed, which could form the bases for future studies on signal processing involving ordered data. These results are summarized below.

The characteristic function of the μ-th largest random variable x'_μ is given in (A-23), while the joint characteristic function of the μ-th and ν-th largest random variables, x'_μ and x'_ν, is given in (A-25), and the characteristic function of difference $x'_\mu - x'_\nu$ is given in (A-33). The joint characteristic function of $x'_{\mu_1},\ldots,x'_{\mu_M}$ is given in (A-34), and then specialized to the joint characteristic function of $x'_1,\ldots,x'_M$ in (A-38).

Some more general cases are undertaken in appendix B. The joint characteristic function of $x'_1,\ldots,x'_N$ for independent but <u>differently</u> distributed exponential random variables is given in (B-12), and then specialized in (B-17) to the case where a subset of M of these variables have one density while the remaining N-M random variables have a different density. This has obvious applications to finding the statistics of sum variable z under hypothesis H_1 rather than H_0.

The further specialization to all N variables having the same density leads to (B-18) for the joint characteristic function. Finally, the characteristic function of a weighted sum of ordered random variables is given by (B-20). This last investigation is aimed at trying to improve the performance of test (28) by replacing M by N, and by weighting the larger data points more heavily. In particular, in appendix C, the maximum deflection of the weighted sum of ordered data is solved in (C-9) and (C-10). However, the deflection is not a complete descriptor of performance, involving no more than second-order moments. Also, (C-10) requires extremely accurate calculation of the means $\{\mu_{1n}\}$ and $\{\mu_{0n}\}$ in order to retain any significant digits in the optimum weights $\{\tilde{w}_n\}$.

Finally, in appendix D, an investigation into distortion of the data, after it has been ordered, is conducted. The motivation behind this study is again to see if improved detection performance can be achieved. For independent identically-distributed random variables with arbitrary probability density function p and arbitrary distortion function h, the characteristic function of the sum of the first M distorted random variables is obtained exactly in (D-10) or (D-11).

The rest of appendix D is devoted to special cases. In particular, the characteristic function of the sum of the M largest random variables in a set of N independent identically-distributed exponential random variables is given by (D-23). On the other hand, when the distorted random variables are squared prior to summation, the pertinent characteristic function is given instead by (D-26) and (D-24).

When these latter two problems are reworked for independent identically-distributed Gaussian random variables rather than exponential random variables, the characteristic function for the linear sum is given by (D-28) and (D-27), while the corresponding result for a sum of squares is given in (D-30) and (D-29).

Finally, if the original random variables $\{x_n\}$ are N independent identically-distributed chi-squared random variables with 2K degrees of freedom, the characteristic function of the sum of the M largest random variables is given by (D-33).

SUM-OF-M-LARGEST PROCESSOR

This processor is characterized in (28) as the sum of the first M random variables of the ordered data set $\{x'_n\}$, which is equivalent to the sum of the M largest random variables of original set $\{x_n\}$ of size N. The decision variable is

$$z = \sum_{n=1}^{M} x'_n \gtrless v \; . \qquad (28)$$

CHARACTERISTIC FUNCTION UNDER H_0

Under hypothesis H_0, when noise alone is present, and data $\{x_n\}$ is composed of N independent identically-distributed exponential random variables, the exact characteristic function of output z is given by (A-11) as the compact closed form

$$f(\xi) = \frac{1}{(1 - i\xi)^{M-1} \prod_{n=M}^{N} \left(1 - i\xi \frac{M}{n}\right)} \; . \qquad (29)$$

A couple of checks on this result are possible. First, for $M = N$, (29) reduces to $(1 - i\xi)^{-N}$, which is obviously correct, since z is then simply the sum of all the original data $\{x_n\}$. On the other hand, for $M = 1$, z is the largest random variable in set $\{x_n\}$, that is, $z = x'_1 = \max\{x_n\}$, and (29) reduces to

$$f_1(\xi) = \frac{1}{\prod_{n=1}^{N} \left(1 - \frac{i\xi}{n}\right)} \; . \qquad (30)$$

To verify this last result, observe that the cumulative distribution function of x_1' under H_0 is $[1 - \exp(-v)]^N$ for threshold $v > 0$, since all the exponential random variables $\{x_n\}$ must stay below v. The corresponding probability density function is $N \exp(-u) [1 - \exp(-u)]^{N-1}$ for $u > 0$, for which the characteristic function (Fourier transform) is just (30). Here, we used result (A-17) with the identifications $\alpha = 1$, $\beta = 1$, and $K = N - 1$.

In this special case of $M = 1$, that is $z = \max\{x_n\}$, the exceedance distribution function of output z under H_1 is also immediately available in closed form. It is given by

$$E_z^1(v) = \Pr(z > v|H_1) = 1 - [1 - \exp(-\underline{a}v)]^{\underline{M}} [1 - \exp(-v)]^{N-\underline{M}}, \quad (31)$$

where we have used (1) and (2) with common signal power $\underline{S}$ and parameter $\underline{a} \equiv (1 + \underline{S})^{-1}$. $\underline{M}$ is the actual number of bins occupied by signal.

Returning now to general results (28) and (29) for arbitrary N and M, the mean of sum z under H_0 is readily found to be

$$\bar{z} = M + \sum_{n=M+1}^{N} \frac{M}{n} . \quad (32)$$

This result is required when we employ the numerical techniques in [3] for accurately and efficiently determining the exceedance distribution function directly from the characteristic function. In this case of hypothesis H_0, we will obtain the false alarm probability directly from characteristic function (29), that is, $P_f = \Pr(z > v|H_0)$, for arbitrary M and N.

FALSE ALARM PROBABILITY

For M = 1, that is, $z = \max\{x_n\}$, the false alarm probability follows directly from (31) by setting signal power $\underline{S} = 0$, that is, $\underline{a} = 1$, thereby getting

$$P_f(M=1) = \Pr(z > v|H_0) = 1 - [1 - \exp(-v)]^N \quad \text{for } v > 0 \ . \qquad (33)$$

Since this result is easily evaluated, no plots are presented here for this special case of M = 1, for the sake of brevity. However, false alarm probability (33) and detection probability (31) will be used later to generate some of the tabular results that follow.

Similarly, for M = N, where z is the sum of all the random variables $\{x_n\}$, no analyses or plots are given here, because they have already been given in [1; pages 21 - 22 and 81 - 90]. Thus, for the most part, attention is confined to $1 < M < N$ here.

In figure 1, the false alarm probability P_f for M = 2 and for $N = 2^k$ with k = 1(1)10 is plotted versus threshold v over a range including P_f values down to 1E-6. Similar sets of false alarm probability plots are presented in figures 2 - 10 as M varies over the values 3, 4, 8, 16, 32, 64, 128, 256, 512, respectively. In every case, the smallest value of N considered is M, because sum z in (28) is only defined for $M \le N$. These results are very accurate, even at the 1E-6 level, because the efficient fast Fourier transform procedure in [3] was employed for going directly from exact characteristic function (29) to the false alarm probability, with insignificant aliasing error.

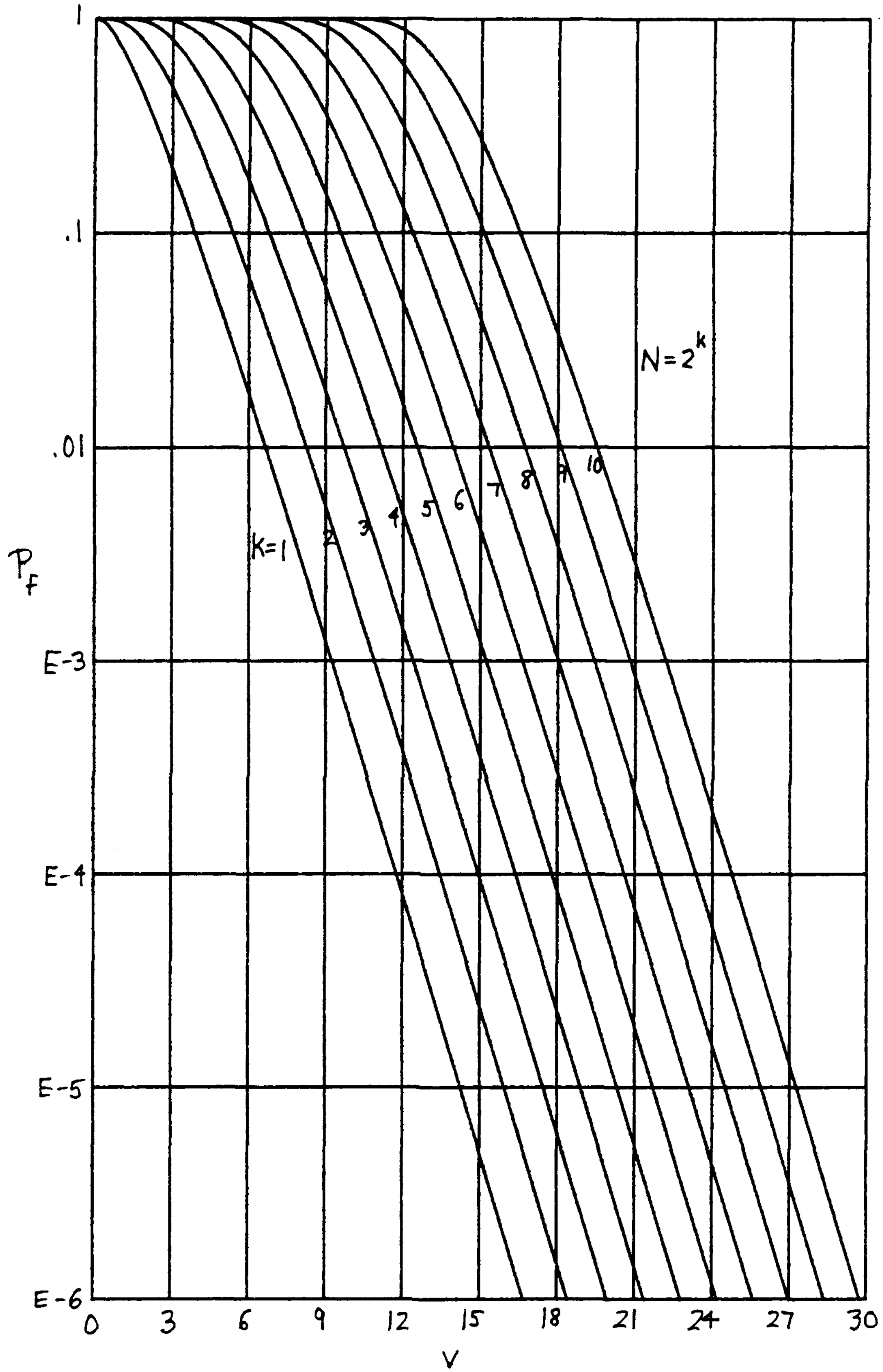

Figure 1. False Alarm Probability for M = 2

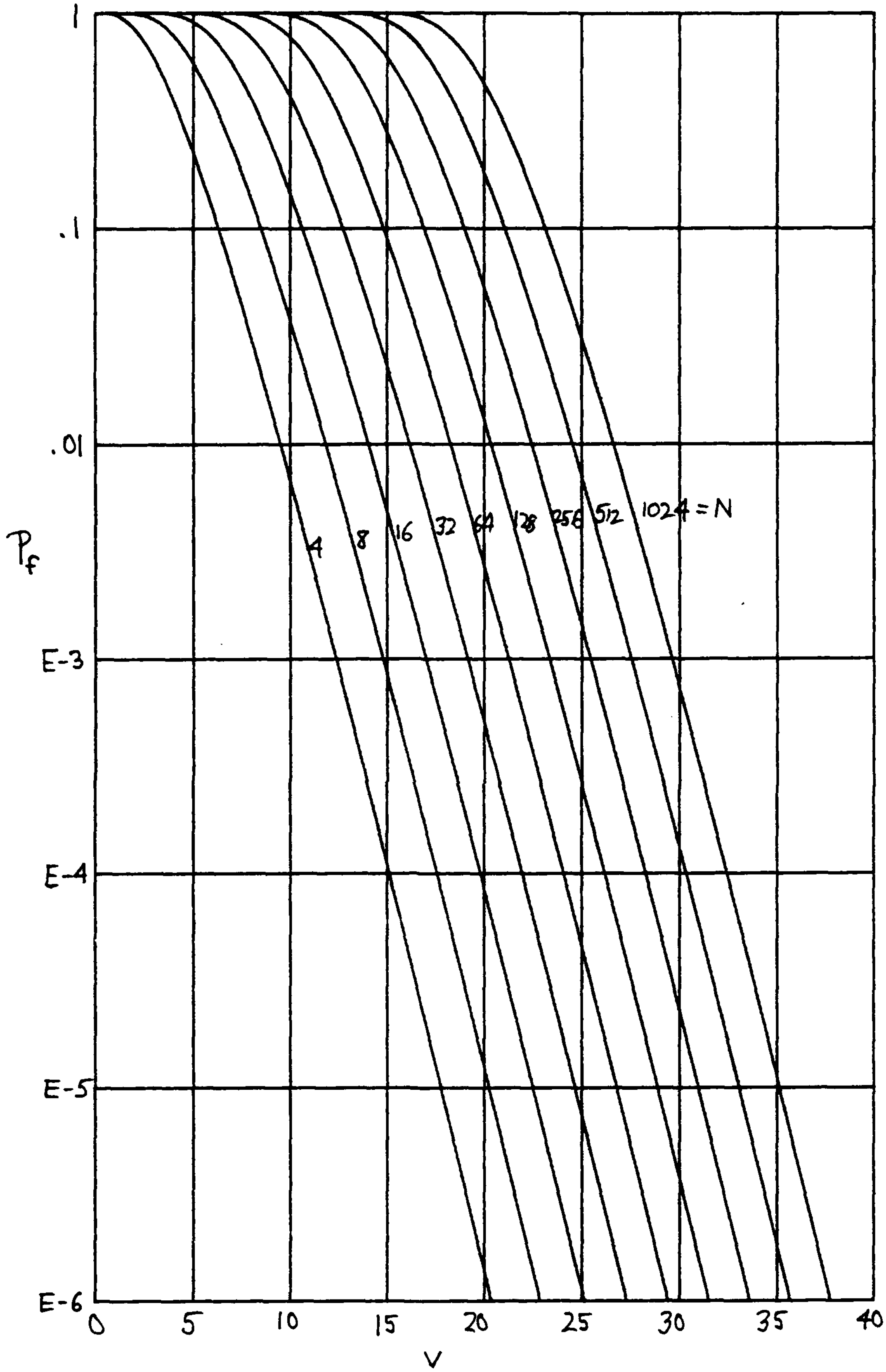

Figure 2. False Alarm Probability for M = 3

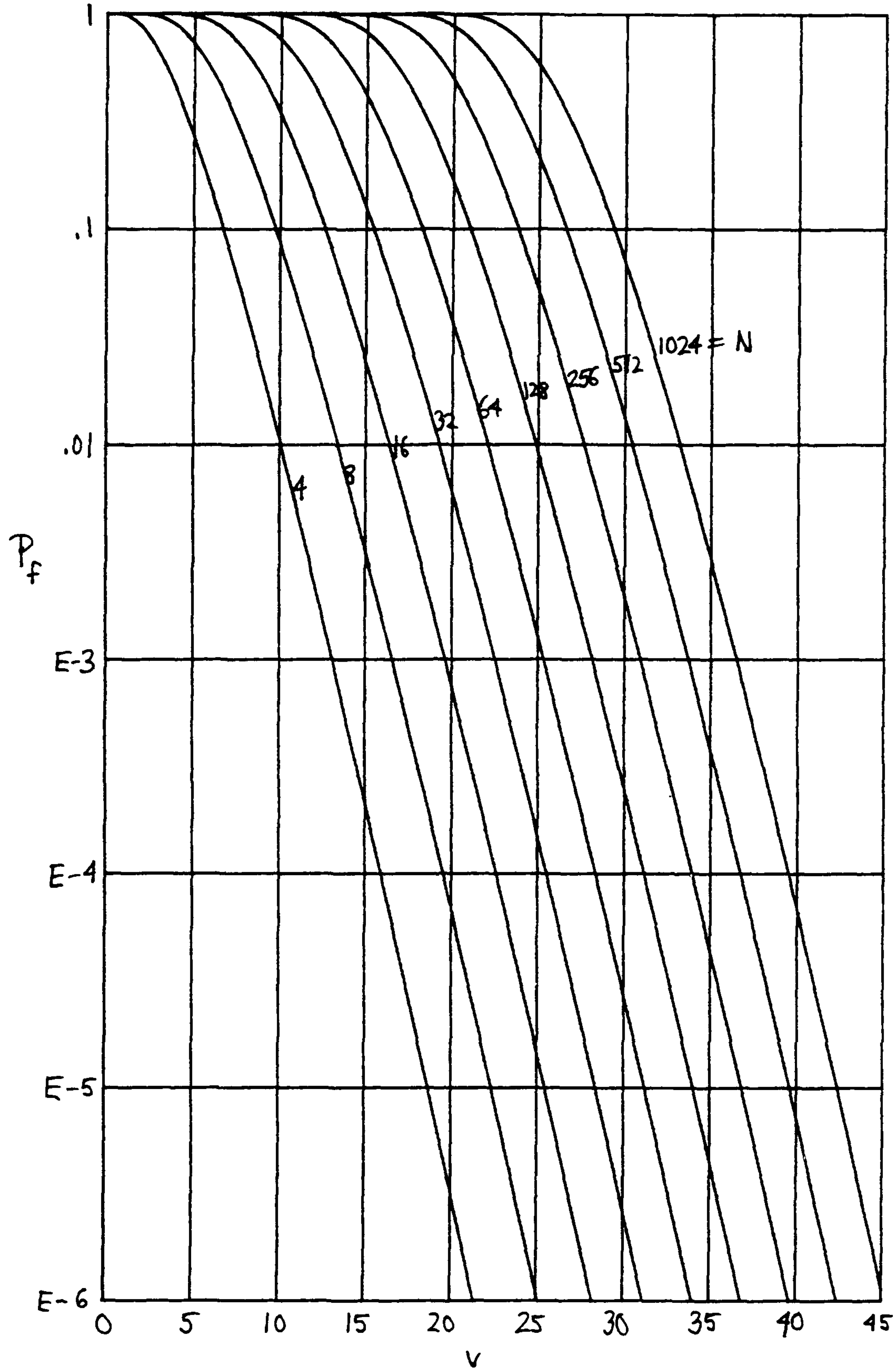

Figure 3. False Alarm Probability for M = 4

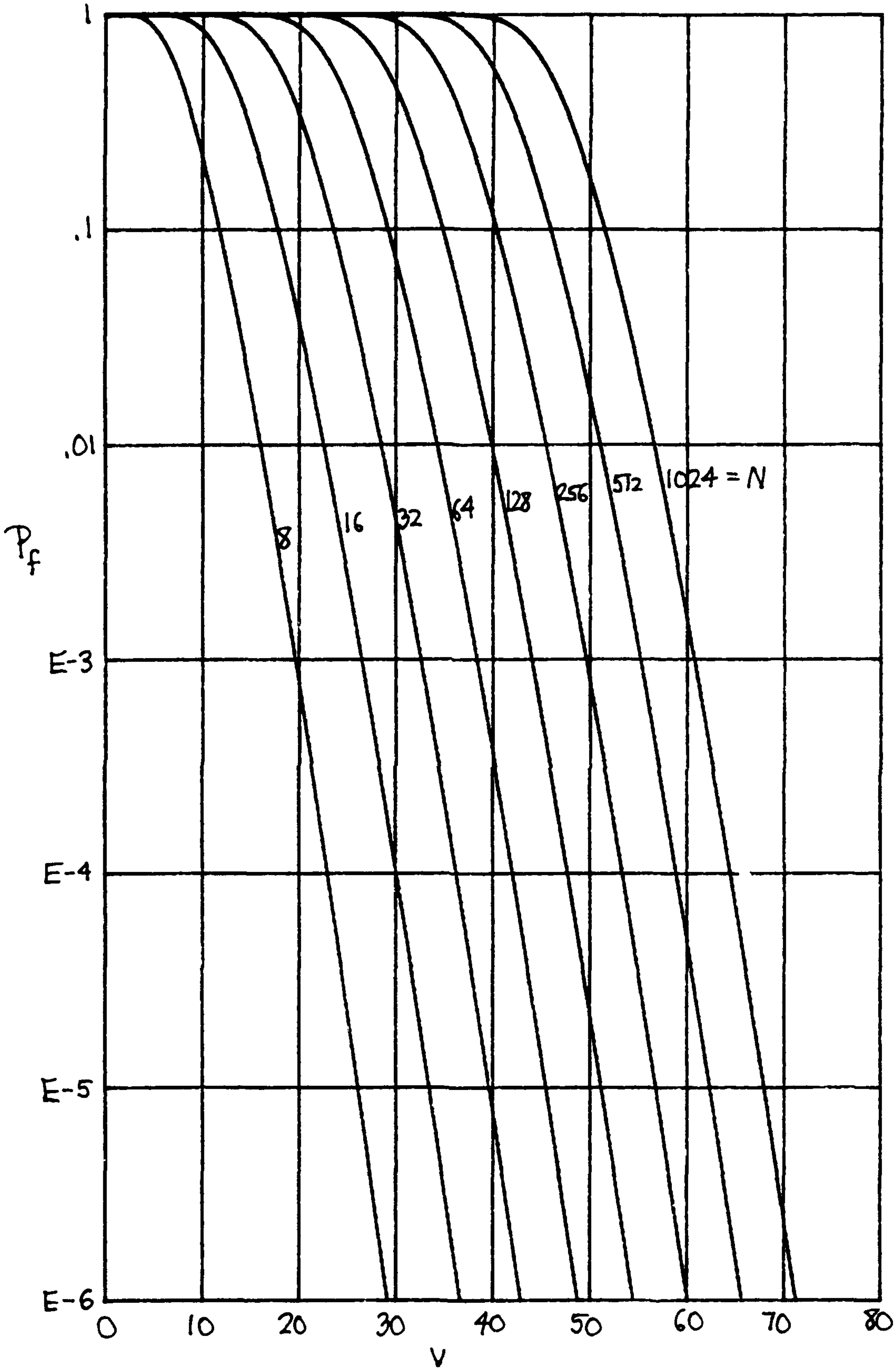

Figure 4. False Alarm Probability for M = 8

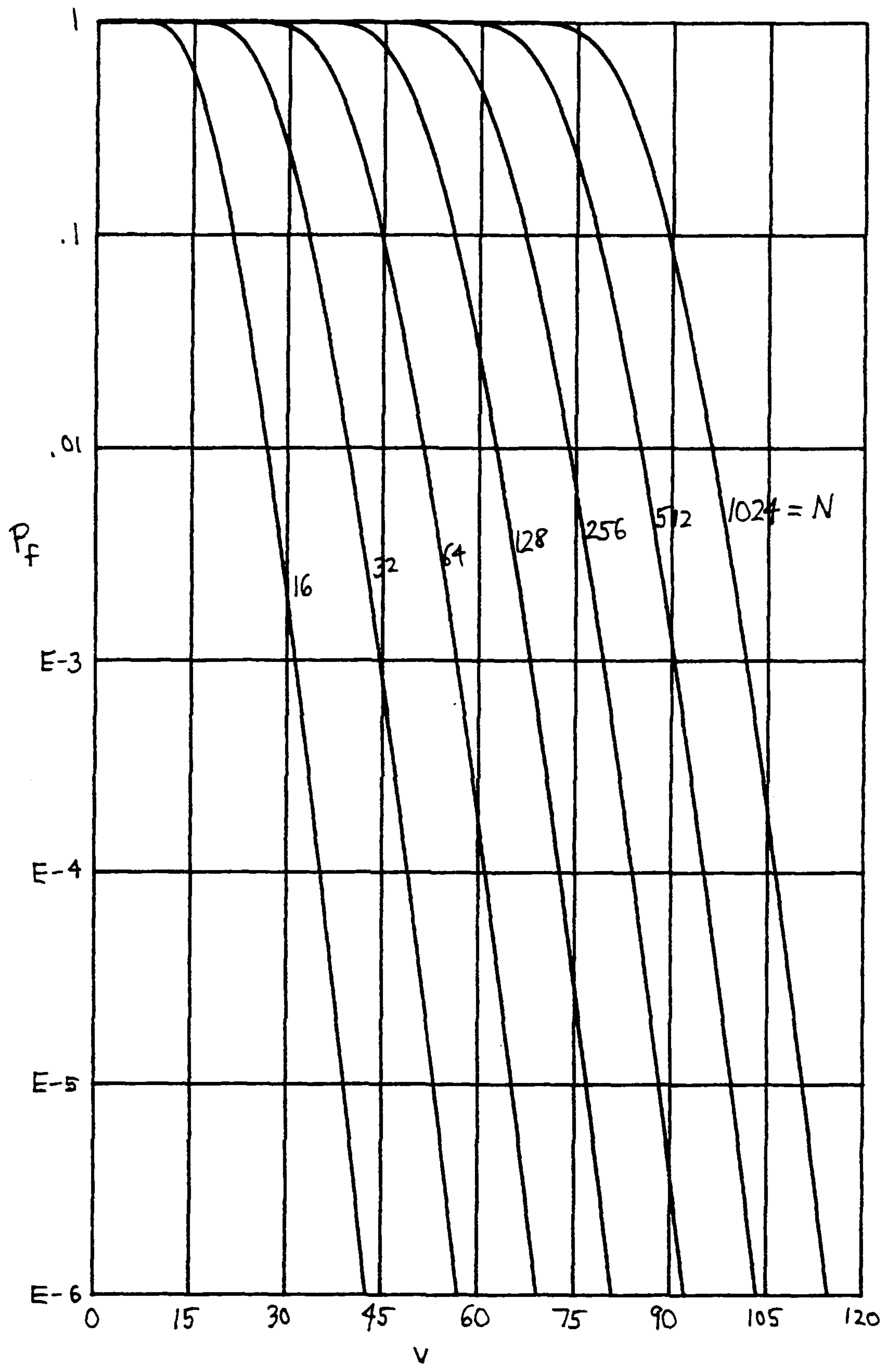

Figure 5. False Alarm Probability for M = 16

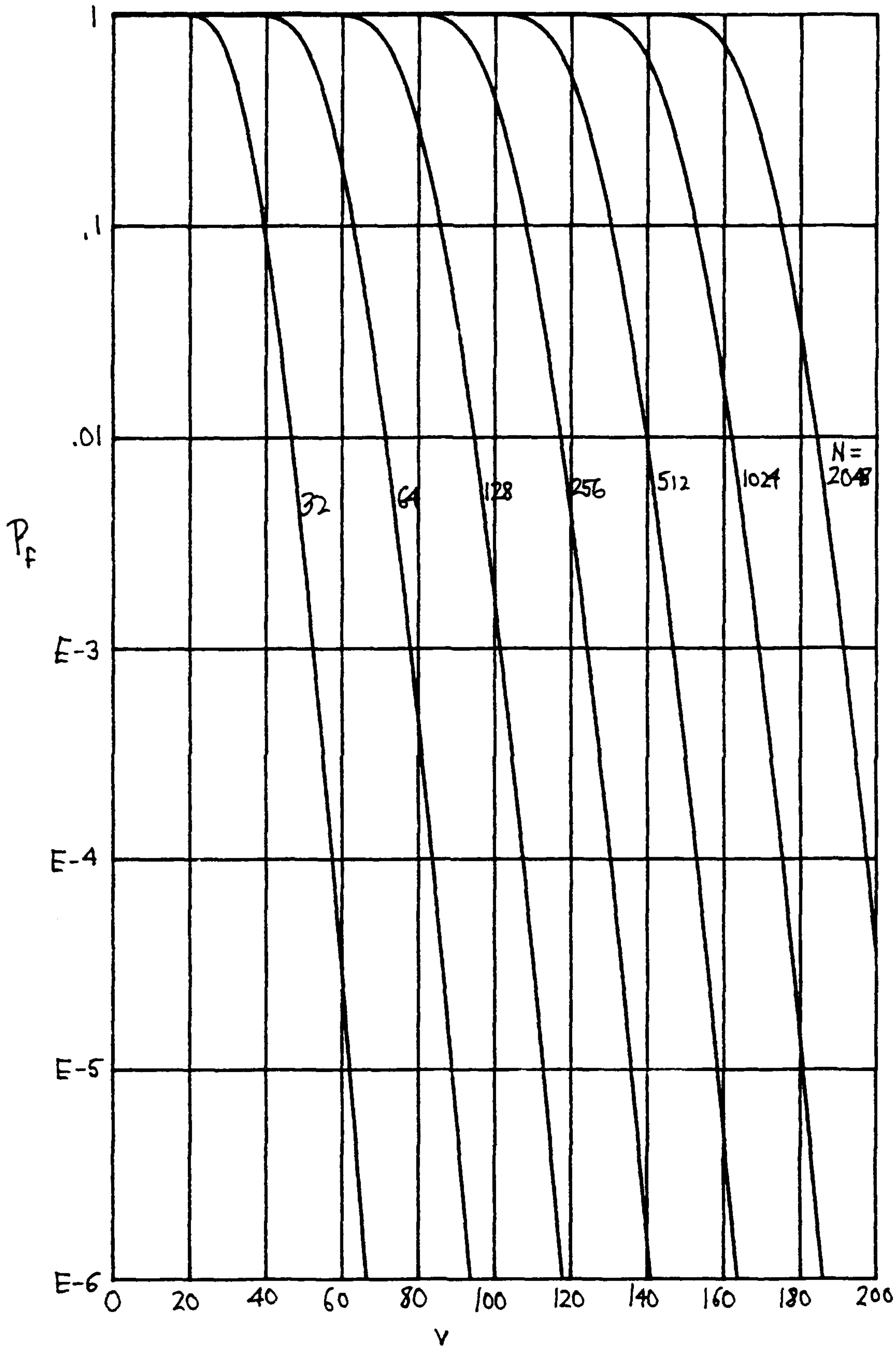

Figure 6. False Alarm Probability for M = 32

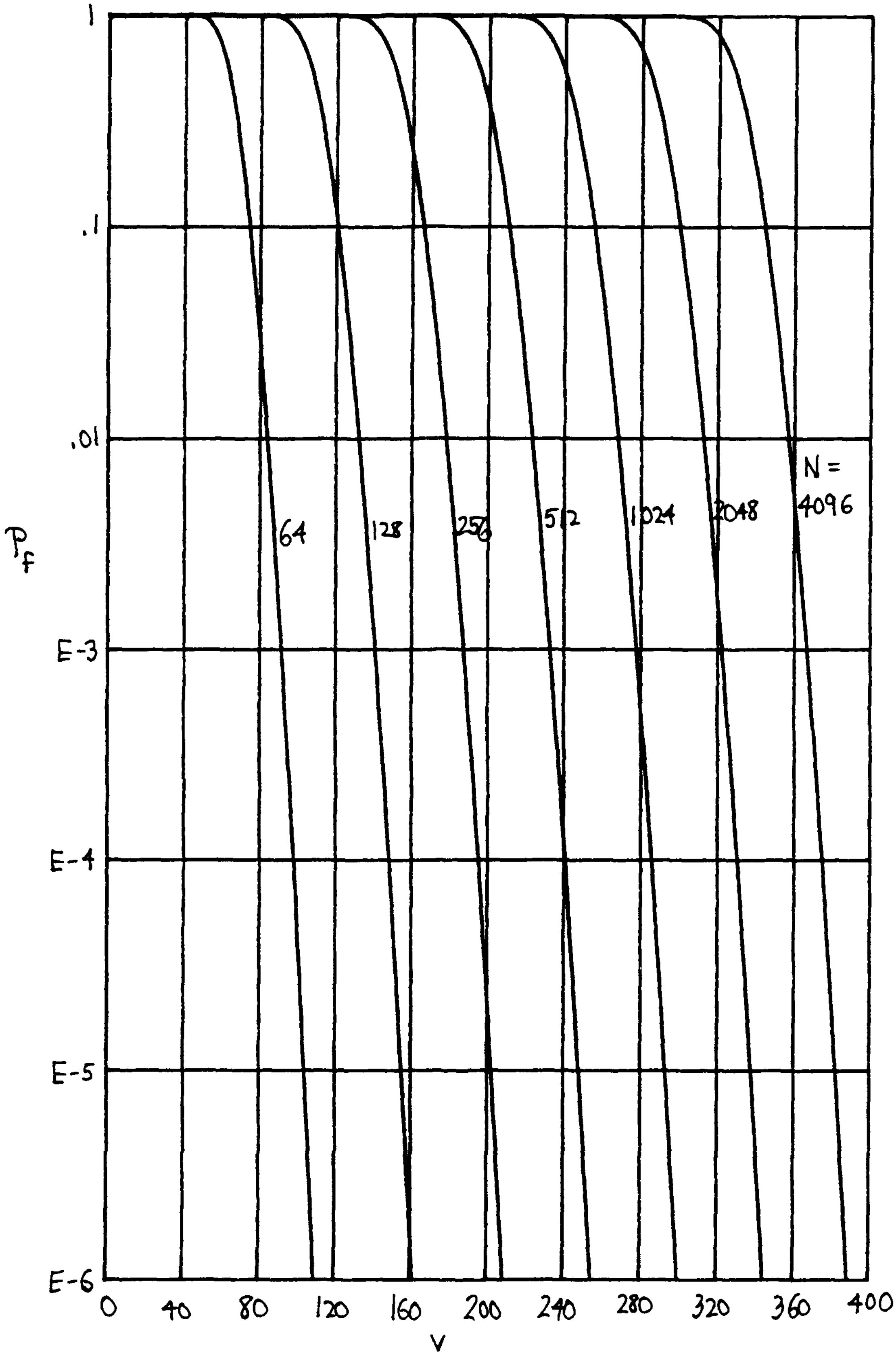

Figure 7. False Alarm Probability for M = 64

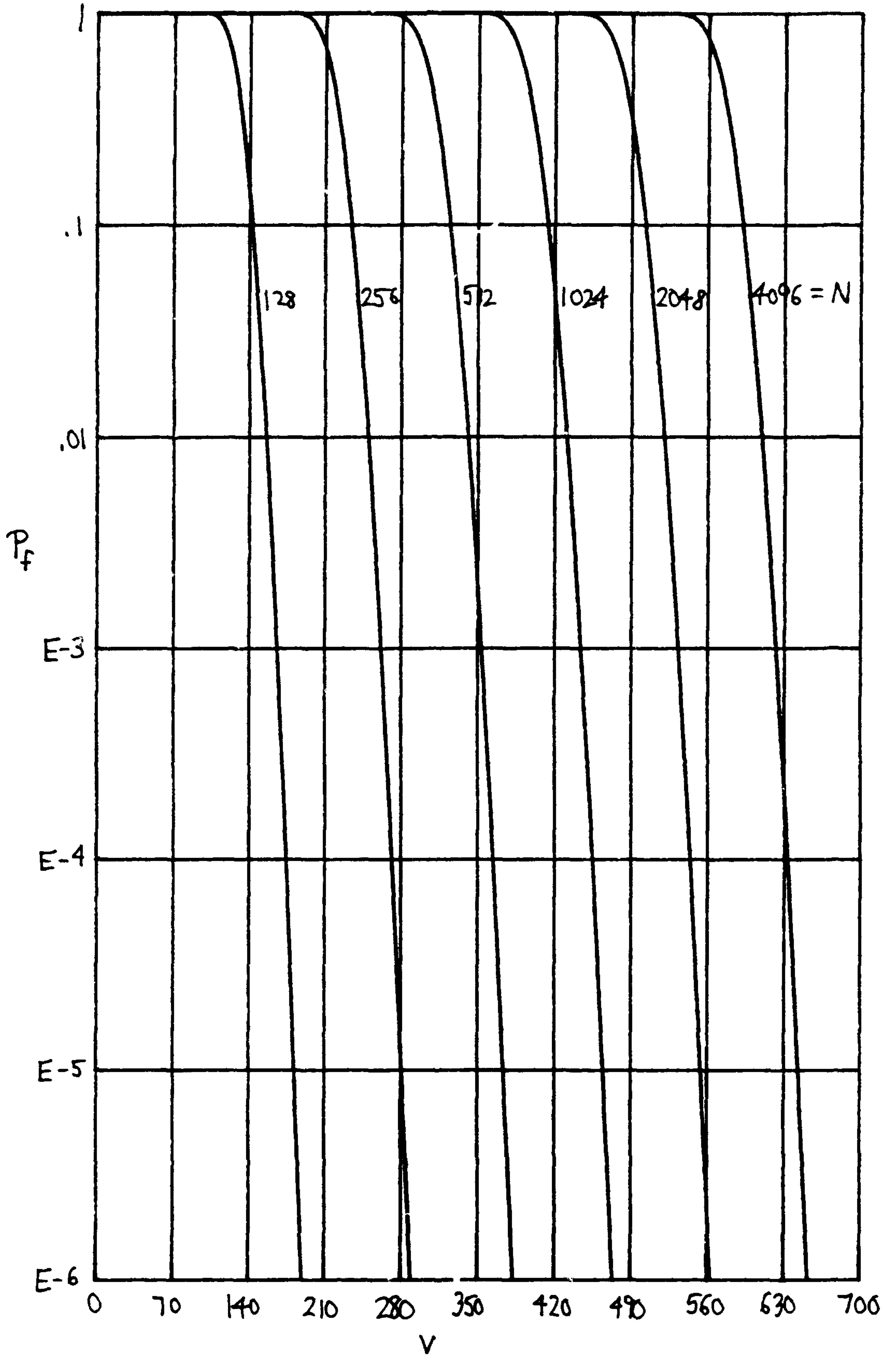

Figure 8. False Alarm Probability for M = 128

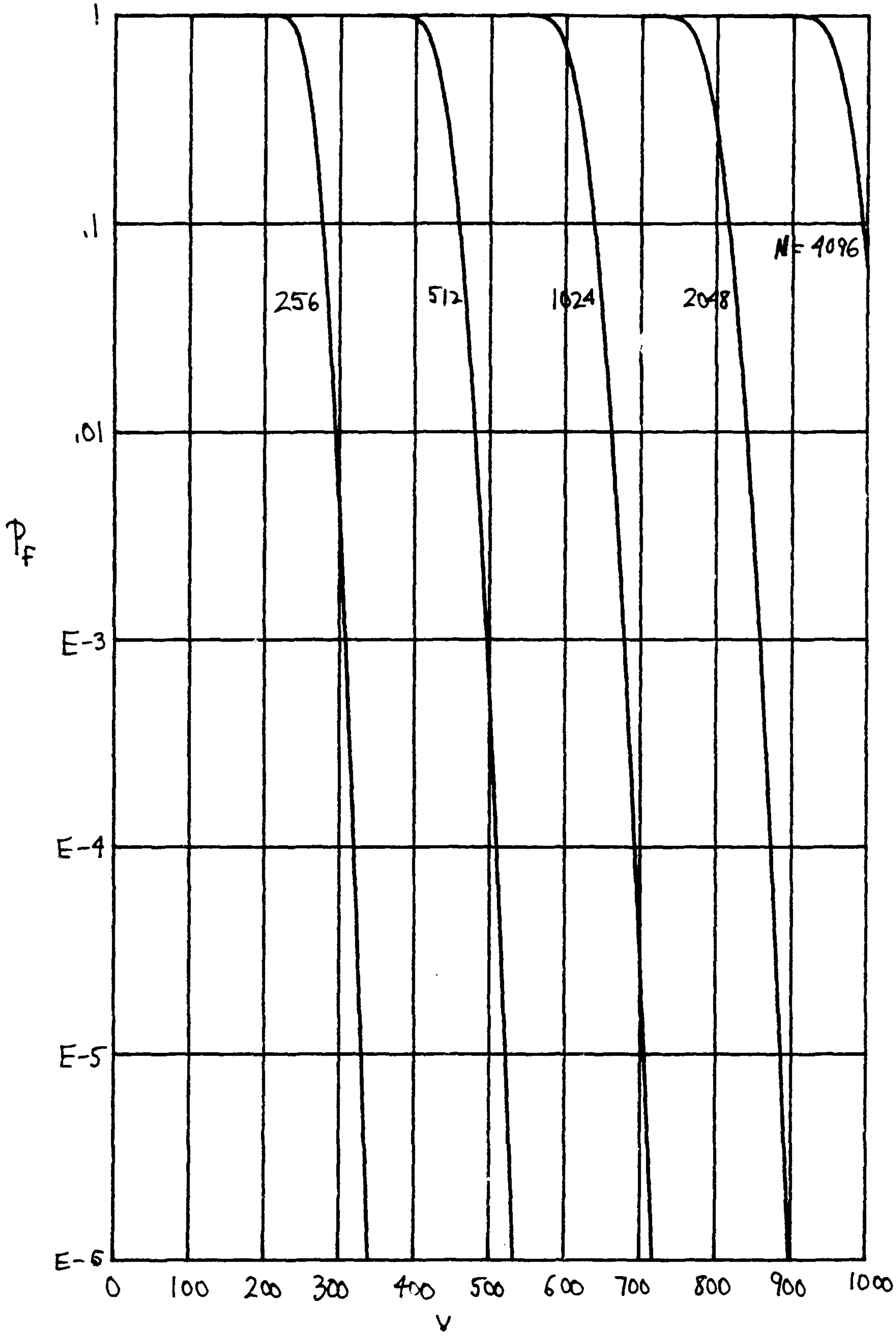

Figure 9. False Alarm Probability for M = 256

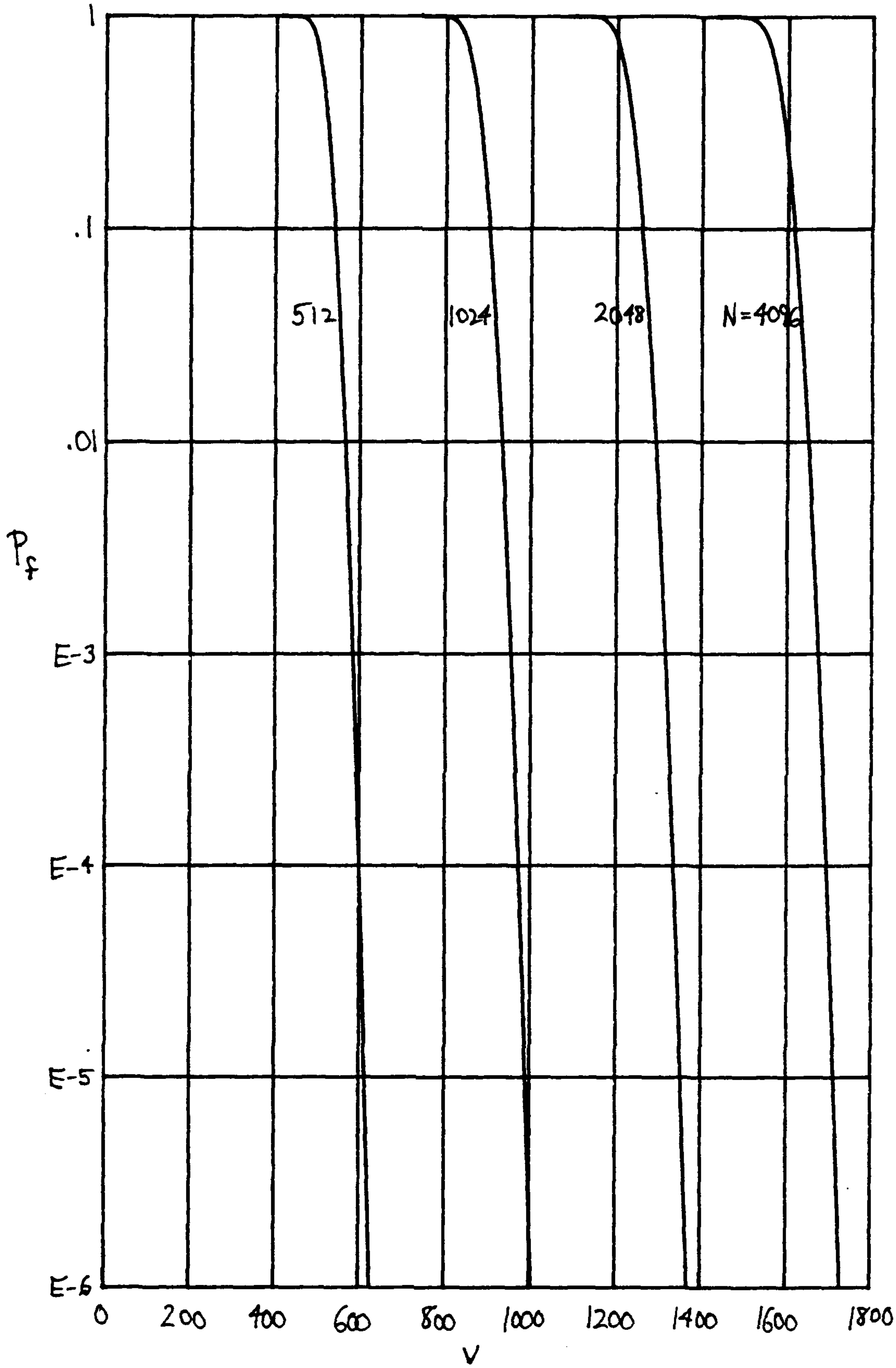

Figure 10. False Alarm Probability for M = 512

DETECTION PROBABILITY

In all the numerical results to follow, the total search space will be taken at the fixed value N = 1024. Then, actual signal size $\underline{M}$ will be allowed to vary over the 12 values

$$\underline{M} = 1, 2, 3, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \tag{34}$$

which allows us to consider signal structures varying all the way from tonals to very broadband processes. At the same time, hypothesized signal size M will vary over the 10 values

$$M = 2, 3, 4, 8, 16, 32, 64, 128, 256, 512, \tag{35}$$

consistent with the theme above that there is no need to present the results for M = 1 or M = N. Thus, we have 120 cases to consider for the detection probability of the sum-of-M-largest processor. This will allow us to extract extensive quantitative results on the degradation suffered by mismatching hypothesized value M to actual (unknown) value $\underline{M}$.

Under hypothesis H_1, when signal is present in $\underline{M}$ bins of the data $\{x_n\}$ leading to sum z in (28), the characteristic function of output z cannot be found in any practically useful closed form. This is true even if the signal powers are all equal to a common value $\underline{S}$ in the $\underline{M}$ occupied bins. This conclusion is based on the analytical results presented in appendix B, especially (B-15) - (B-17) coupled with (B-3) - (B-4). A numerical example for N = 4 and $\underline{M}$ = 2 in (B-24) and sequel illustrates the severe complexity of the general result.

This inability to analytically obtain the characteristic function of z under H_1 has forced us to simulate the detection probability results. An example of the resultant receiver operating characteristics is given in figure 11, where 8300 independent trials were used for the ten different values of signal power $\underline{S}$ considered. The abscissa value, false alarm probability P_f, of each point on these plots is exact, having been obtained by means of (29). Therefore, we are able to reliably carry these curves down to the small values for P_f around 1E-6.

However, the ordinate value, detection probability P_d, has jitter (random perturbations) in it due to the limited number of independent trials, namely 8300 for this particular example. Since we are usually interested in P_d values in the neighborhood of .5 to .9, this number of trials is sufficient to generate an accurate receiver operating characteristic in the range of interest. The jitter is most noticeable in the unimportant upper right-hand corner of the figure where large P_d values near .99 are being estimated, and where P_f is too large to be practically useful.

The totality of 120 receiver operating characteristics generated by the cases listed in (34) and (35) are collected together in appendix E. The number of trials varies widely, from 5600 to 35000 trials, depending upon the time that happened to be available for the particular run. The signal power per bin, $\underline{S}$ in dB, always varies over a range sufficient to cover the useful values of false alarm and detection probabilities.

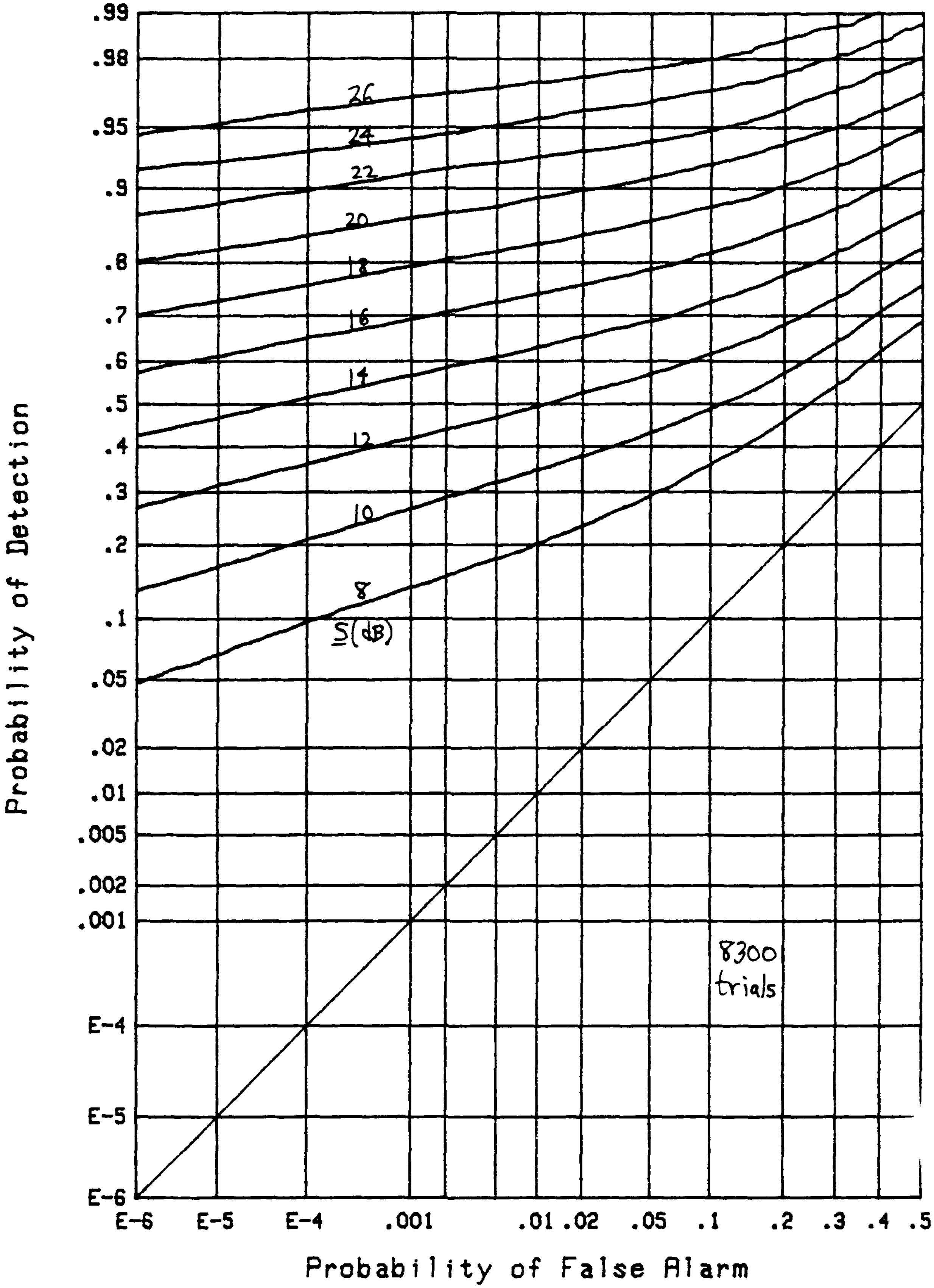

Figure 11. Operating Characteristic for N = 1024, M = 1, M = 2

REQUIRED SIGNAL-TO-NOISE RATIO

The voluminous compilation (120 plots) of receiver operating characteristics in appendix E forces us to condense this information for easier interpretation and accessibility. To accomplish this, we define a low-quality operating point $P_f = 1E-3$, $P_d = .5$ and a high-quality operating point $P_f = 1E-6$, $P_d = .9$. We then read off the curves in appendix E the values of signal power $\underline{S}$(dB) which are required to realize these two levels of performance. These results are tabulated in tables 1 and 2 for the low-quality and high-quality operating points, respectively, for $\underline{M}$ and M both ranging over the full set of values 1, 2, 3, 4, 8, 16, 32, 64, 128, 256, 512, 1024.

It is immediately seen that the best value of M, for minimum $\underline{S}$, is not necessarily $\underline{M}$, although the discrepancies are small. For example, if $\underline{M} = 16$, the best value for M is smaller than $\underline{M}$, being in the range 4 to 8. On the other hand, if $\underline{M} = 256$, the best M is larger than $\underline{M}$, namely 512. The explanation for this effect is given in appendix F; it has to do with the fact that the generalized likelihood ratio processor is not necessarily optimum.

The results in tables 1 and 2, are plotted in figures 12 and 13, respectively, with one modification; the ordinates in the figures are the total signal power required, $\underline{M}\ \underline{S}$ in dB, rather than just the bin power, $\underline{S}$. The total quantity is more meaningful and it condenses the range of ordinate values to a more manageable regime.

Table 1. Required $\underline{S}$(dB) for Sum-of-M-Largest Processor, N = 1024, $P_f = 10^{-3}$, $P_d = .5$

$\underline{M}$ \ M	1	2	3	4	8	16	32	64	128	256	512	1024
1	12.77	13.1	13.4	13.6	14.2	15.1	16.3	17.4	18.6	19.9	20.9	21.53
2	10.11	10.05	10.3	10.45	11.0	11.7	12.6	13.7	14.8	16.05	17.1	17.71
3	8.90	8.75	8.9	9.0	9.45	10.05	10.9	11.9	12.95	14.05	15.1	15.70
4	8.14	7.95	8.0	8.1	8.45	9.0	9.7	10.65	11.65	12.7	13.75	14.34
8	6.59	6.25	6.2	6.2	6.35	6.75	7.25	7.9	8.75	9.7	10.6	11.17
16	5.28	4.7	4.55	4.5	4.5	4.7	5.0	5.4	6.05	6.8	7.6	8.09
32	4.15	3.45	3.2	3.1	2.95	2.9	2.95	3.2	3.55	4.05	4.65	5.05
64	3.12	2.2	1.9	1.7	1.3	1.15	1.05	1.05	1.15	1.35	1.7	2.03
128	2.17	1.1	0.7	0.35	-0.2	-0.6	-0.85	-1.1	-1.25	-1.3	-1.2	-0.98
256	1.27	0.0	-0.6	-0.9	-1.7	-2.3	-2.85	-3.3	-3.7	-3.95	-4.1	-3.99
512	0.39	-1.0	-1.75	-2.25	-3.25	-4.2	-4.95	-5.7	-6.25	-6.7	-7.0	-7.00
1024	-0.48	-2.2	-3.05	-3.7	-5.0	-6.2	-7.25	-8.2	-8.95	-9.55	-9.95	-10.01

Table 2. Required $\underline{S}$(dB) for Sum-of-M-Largest Processor, $N = 1024$, $P_f = 10^{-6}$, $P_d = .9$

$\underline{M}$ \ M	1	2	3	4	8	16	32	64	128	256	512	1024
1	22.92	23.2	23.4	23.7	24.5	25.4	26.5	27.7	28.8	30.1	31.2	31.81
2	17.29	17.0	17.3	17.5	18.1	18.8	19.8	20.85	22.0	23.2	24.3	24.85
3	15.09	14.6	14.6	14.75	15.25	15.9	16.8	17.75	18.9	20.05	21.15	21.73
4	13.82	13.1	13.1	13.2	13.6	14.2	15.0	15.95	17.1	18.1	19.2	19.77
8	11.45	10.5	10.3	10.25	10.4	10.85	11.45	12.2	13.15	14.1	15.15	15.61
16	9.70	8.5	8.1	7.95	7.85	8.0	8.4	8.95	9.75	10.55	11.4	11.94
32	8.31	7.0	6.45	6.2	5.85	5.8	5.9	6.25	6.75	7.35	8.1	8.56
64	7.17	5.7	5.1	4.75	4.1	3.85	3.8	3.85	4.05	4.45	4.95	5.34
128	6.18	4.6	3.8	3.35	2.65	2.05	1.8	1.6	1.6	1.7	1.95	2.22
256	5.31	3.6	2.75	2.25	1.25	0.5	-0.15	-0.6	-0.85	-1.0	-1.0	-0.85
512	4.53	2.7	1.7	1.15	-0.1	-1.15	-2.0	-2.75	-3.25	-3.7	-3.9	-3.89
1024	3.81	1.8	0.8	0.1	-1.5	-2.8	-3.95	-4.95	-5.8	-6.5	-6.8	-6.91

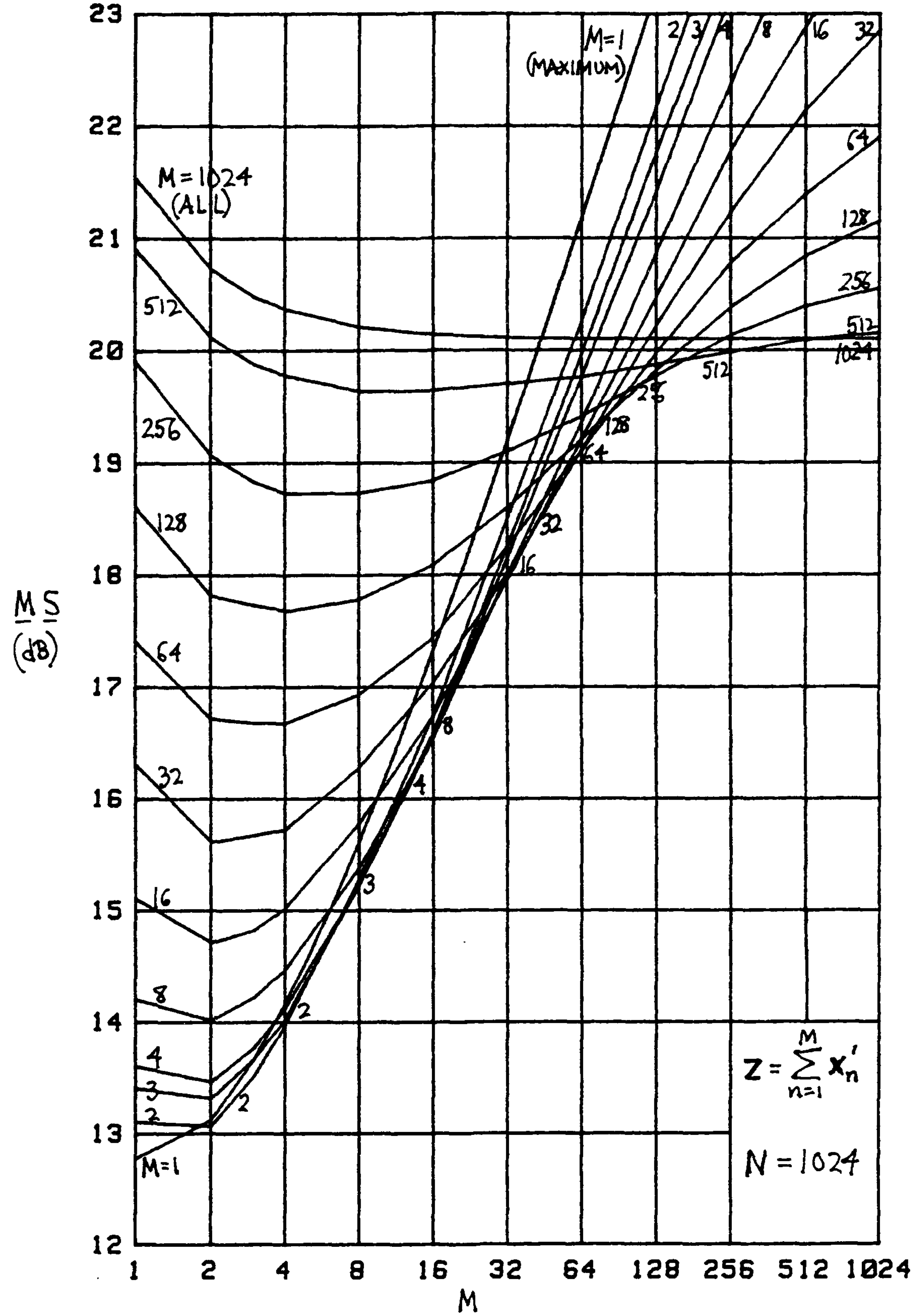

Figure 12. Required Total Signal Power for P_f = 1E-3, P_d = .5

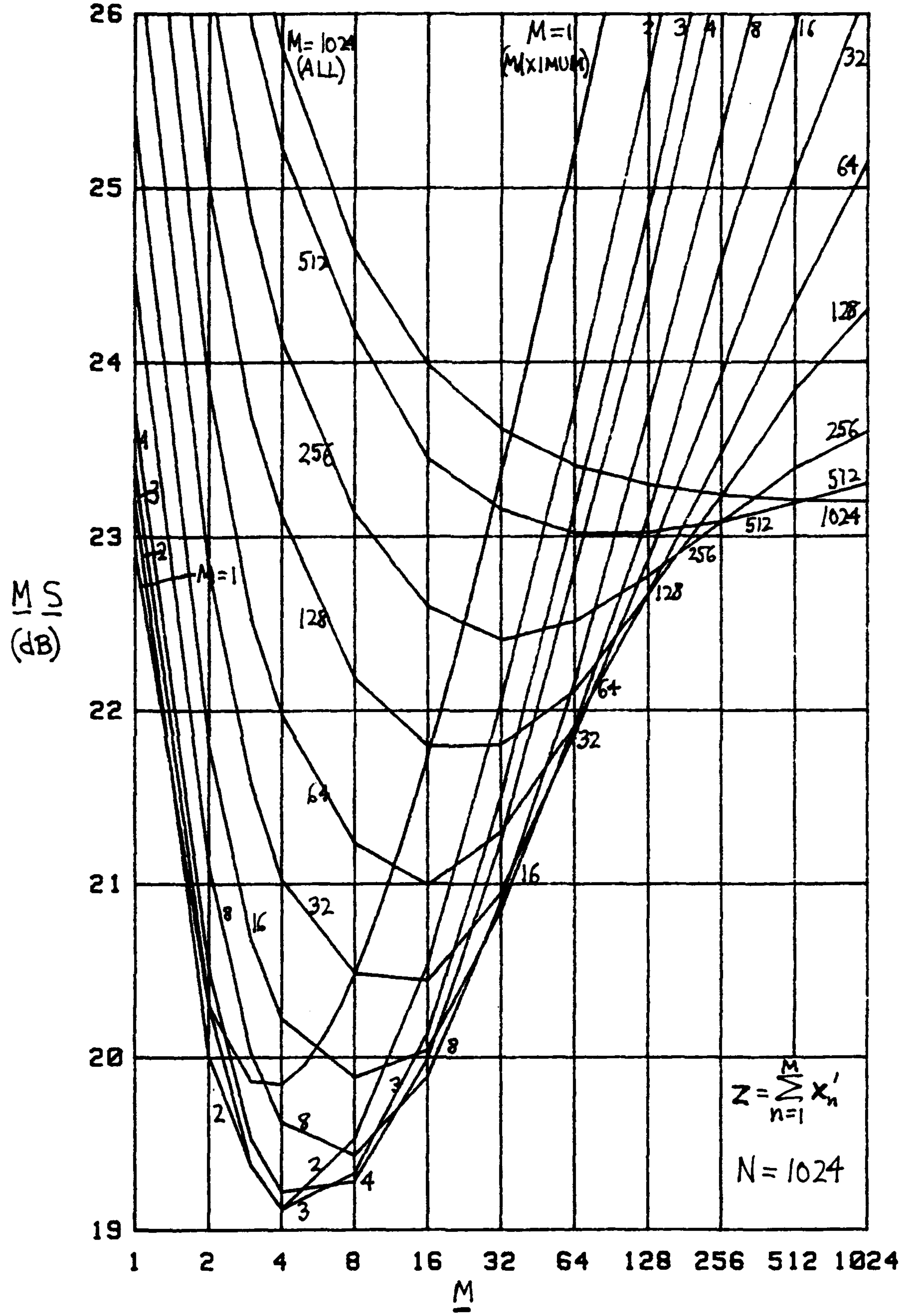

Figure 13. Required Total Signal Power for P_f = 1E-6, P_d = .9

Numerous observations can be made from these two figures. We confine the following numerical examples to figure 12. When $\underline{M} = N = 1024$, the optimum value of M is also 1024, although the loss in using $M = 512$ is less than .1 dB. However, if we continue to use $M = 1024$ when $\underline{M}$ has been decreased significantly below 1024, losses around 9 dB, relative to using $M = 1$, will be incurred when $\underline{M} = 1$ is reached.

On the other hand, suppose we attempt to always use $M = 1$ regardless of the true value $\underline{M}$. Although this selection is satisfactory for $\underline{M}$ less than 32 approximately, it will run into losses greater than 10 dB when the actual value $\underline{M}$ approaches 1024.

This suggests that if nothing whatsoever is known about $\underline{M}$, a compromise value of $M = 32$ might be adopted. The loss when $\underline{M}$ is actually 1024 is then 2.5 dB, whereas the loss when $\underline{M}$ is actually 1 is 3.5 dB. For intermediate values of $\underline{M}$, say from 16 to 64, the consistent use of $M = 32$ is nearly optimum within this class of processors. Thus, if there is some partial information available about the range of $\underline{M}$ values to be encountered, and if this range is narrow enough, figures 12 and 13 indicate what the good choices of M are and the degree of loss caused by mismatch.

Figures 12 and 13 bring out one of the significant drawbacks of the sum-of-M-largest processor, namely that a good choice of upper limit M in sum (28) cannot be made without some knowledge about $\underline{M}$, the number of bins occupied by signal. It also indicates that further study into the determination of a more robust class of processors is warranted and required.

COMPARISON WITH MODIFIED GENERALIZED LIKELIHOOD RATIO PROCESSOR

The modified generalized likelihood ratio (MGLR) processor [1] was derived on the basis of an unknown number of occupied signal bins, $\underline{M}$; in fact, the average signal strength $\underline{S}_n$ was estimated in each and every one of the N bins. However, during data processing, if the MGLR processor uses the best value of its breakpoint x_o for the actual current value of $\underline{M}$, the MGLR processor has effectively been given knowledge of $\underline{M}$. Also, since the best breakpoint value of x_o is typically large for small $\underline{M}/N$, the small-input square-law suppression of nonlinearity

$$g_o(x) \equiv \begin{Bmatrix} x - 1 - \ln(x) & \text{for } x \geq x_o \\ 0 & \text{for } x < x_o \end{Bmatrix} \tag{36}$$

is never encountered, since the nonlinearity is virtually linear for $x > x_o$ in this situation.

Thus, when the MGLR processor is effectively using knowledge of small values $\underline{M}$, it is linearly processing only the largest members of observation $\{x_n\}$. This is exactly what the GLR processor considered here does, except that the number of *contributors to the GLR processor* output is always exactly M (regardless of $\underline{M}$), whereas the MGLR processor output number of contributors fluctuates, depending on the actual input data $\{x_n\}$, N, $\underline{S}$, and $\underline{M}$.

This means that we can expect the best performers in the MGLR class of processors to have comparable performance to the best performers in the sum-of-M-largest class of processors. This is borne out by the results in figures 14 and 15, which are extracted from the receiver operating characteristics for the MGLR processor in [1]. The lower envelopes of figures 12 and 14 are very close over their common range from $\underline{M} = 8$ to $\underline{M} = 256$; similarly, the lower envelopes of figures 13 and 15 are fairly close to each other.

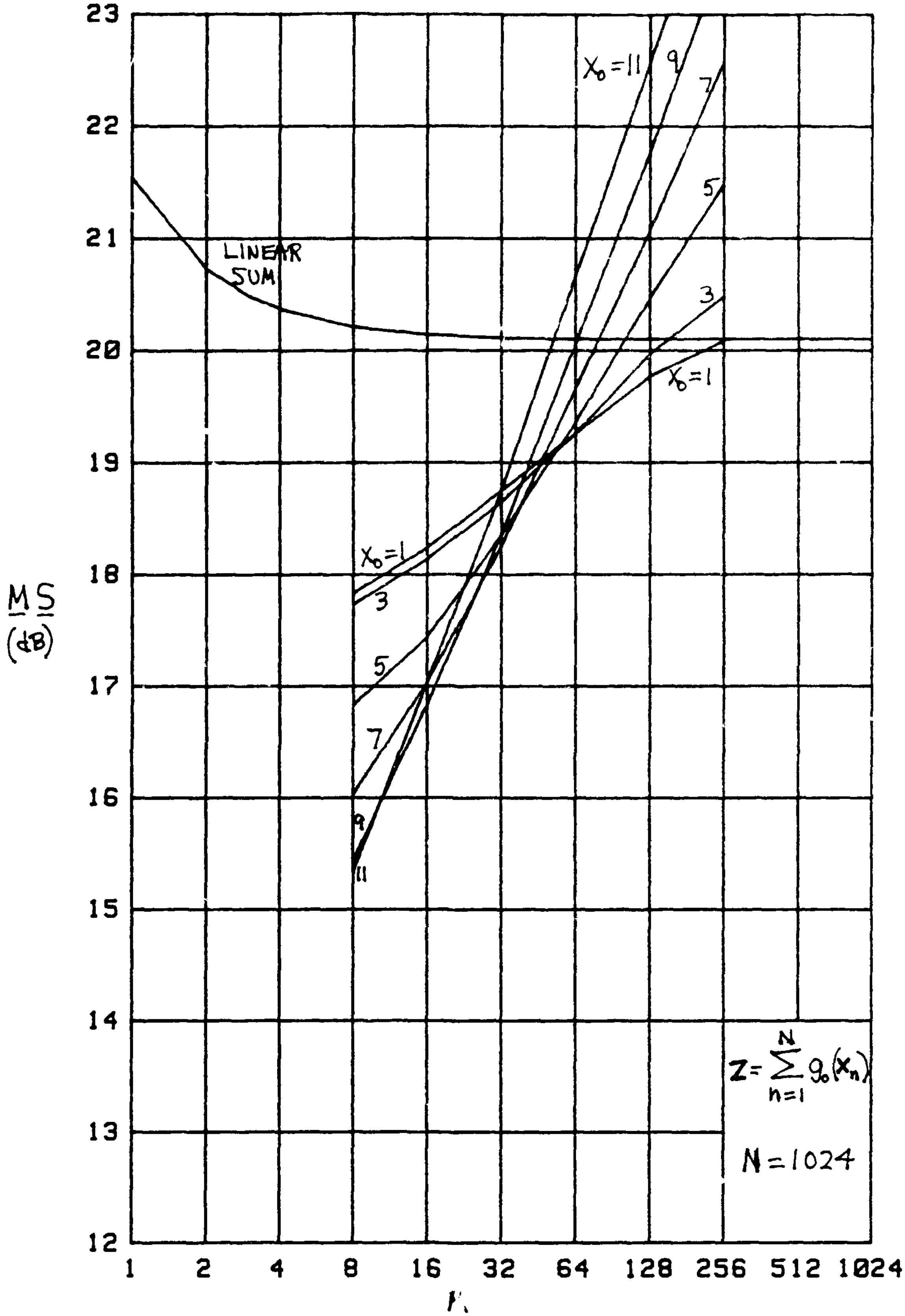

Figure 14. MGLR Total Signal Power for P_f = 1E-3, P_d = .5

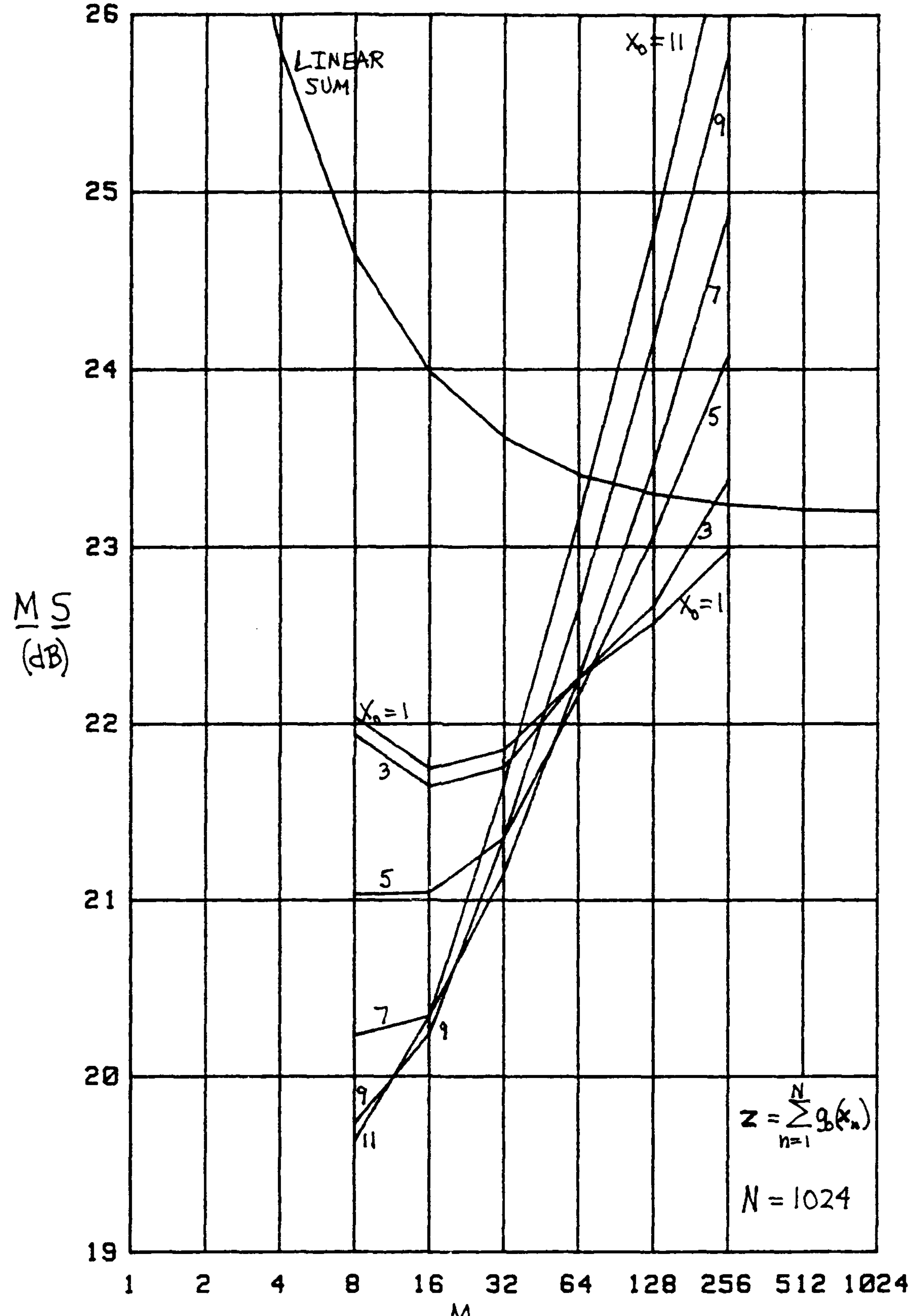

Figure 15. MGLR Total Signal Power for P_f = 1E-6, P_d = .9

SUMMARY

The receiver operating characteristics of the sum-of-M-largest processor have been determined for a wide variety of values of $\underline{M}$, the number of bins occupied by signal, and M, the hypothesized number of occupied bins, for total search size N = 1024. The false alarm probability was very accurately evaluated by using the exact characteristic function of the decision variable under H_0; the detection probability was determined by simulations, each averaging about 10000 independent trials.

The amount of loss associated with mismatch between M and $\underline{M}$ can be significant. However, for N = 1024, if one compromises on and uses the intermediate value of M = 32, regardless of the true (unknown) value of $\underline{M}$, the loss is no more than about 3 dB. If some partial knowledge about the range of values of $\underline{M}$ is available, a mid-range choice of M can be made, with reduced losses in mismatch. Quantitative assessment of these losses are possible from the results given here.

Some new results for the characteristic function of a weighted sum of ordered data have been derived, and then used for false alarm probability calculations. Extensions to the characteristic function of the sum of distorted ordered data, as well as to some joint characteristic functions, have also been accomplished, although they were not used to obtain the numerical results here.

APPENDIX A. CHARACTERISTIC FUNCTION OF SUM OF M LARGEST RANDOM VARIABLES OF A SET OF N EXPONENTIAL RANDOM VARIABLES

Consider N independent identically-distributed random variables (RVs) $\{x_n\}$ with common continuous probability density function p and cumulative distribution function C. Then, the probability that the μ-th largest random variable x'_μ lies in interval u,u+du is given by [4; page 370]

$$g_\mu(u)\ du = \Pr\{N-\mu \text{ RVs} < u;\ \mu-1 \text{ RVs} > u+du;\ 1 \text{ RV in } du\} =$$

$$= N \binom{N-1}{\mu-1} C(u)^{N-\mu}\,[1 - C(u)]^{\mu-1}\,p(u)\ du\ . \qquad \text{(A-1)}$$

The probability density function of the μ-th largest random variable is g_μ.

More generally, consider two intervals centered on the values u and v, where u > v, and let integers $\mu < \nu$. Then, the probability that the μ-th largest random variable x'_μ lies in u,u+du and that the ν-th largest random variable x'_ν lies in v,v+dv is

$$g_{\mu\nu}(u,v)\ du\ dv =$$

$$= \Pr\{\mu-1 \text{ RVs } \varepsilon\ (u,\infty);\ \nu-\mu-1 \text{ RVs } \varepsilon\ (v,u);\ N-\nu \text{ RVs } \varepsilon\ (-\infty,v);$$
$$1 \text{ RV } \varepsilon\ (u,u+du);\ 1 \text{ RV } \varepsilon\ (v,v+dv)\} =$$

$$= N\,(N-1) \binom{N-2}{\mu-1} \binom{N-\mu-1}{N-\nu} [1 - C(u)]^{\mu-1}\,[C(u) - C(v)]^{\nu-\mu-1} \times$$
$$\times\ C(v)^{N-\nu}\,p(u)\,p(v)\ du\ dv\ . \qquad \text{(A-2)}$$

The joint probability density function of the μ-th and ν-th largest random variables of set $\{x_n\}$ is $g_{\mu\nu}$.

Continuing in a similar fashion, the joint probability density function of the M largest random variables, out of the set of N independent identically-distributed random variables $\{x_n\}$, is, for values $u_1 > u_2 > \cdots > u_M$,

$$g(u_1,u_2,\ldots,u_M) = N(N-1)\cdots(N-M+1)\ C(u_M)^{N-M} \prod_{m=1}^{M} p(u_m) \ . \quad \text{(A-3)}$$

We are interested in the characteristic function of the sum s of the M largest random variables in set $\{x_n\}$. Notice that the components $\{x_n'\}$ of this sum consist of M <u>statistically</u>-<u>dependent</u> <u>non</u>-<u>Gaussian</u> <u>non</u>-<u>identically</u> distributed random variables. The desired characteristic function is given by

$$f_s(\xi) = \overline{\exp(i\xi s)} = \overline{\exp(i\xi[x_1' + \cdots + x_M'])} =$$

$$= \iint\cdots\int du_1\ du_2\cdots\ du_M\ g(u_1,u_2,\ldots,u_M)\ \exp(i\xi[u_1 + \cdots + u_M]) =$$

$$= \int_{-\infty}^{\infty} du_1\ \exp(i\xi u_1) \int_{-\infty}^{u_1} du_2\ \exp(i\xi u_2)\cdots \int_{-\infty}^{u_{M-1}} du_M\ \exp(i\xi u_M)\ g(u_1,\ldots,u_M)$$

$$= \int_{-\infty}^{\infty} du_M\ \exp(i\xi u_M)\cdots\int_{u_3}^{\infty} du_2\ \exp(i\xi u_2) \int_{u_2}^{\infty} du_1\ \exp(i\xi u_1)\ g(u_1,\ldots,u_M). \quad \text{(A-4)}$$

The last form in (A-4) is more useful, because of the way that cumulative $C(u_M)$ appears in the density $g(u_1,\ldots,u_M)$; namely, the integration on $C(u_M)$ is deferred to the last integral in (A-4).

In general, substitution of (A-3) in (A-4) results in an intractable multiple integral. However, for the case of an exponential probability density function for original random variables $\{x_n\}$,

$$p(u) = \exp(-u) \quad \text{for } u > 0 \ , \tag{A-5}$$

all the manipulations can be carried out in closed form. To demonstrate this, substitute (A-3) and (A-5) into (A-4) to obtain

$$f_s(\xi) = F \int_0^\infty du_M \exp(-zu_M) \, [1 - \exp(-u_M)]^{N-M} \int_{u_M}^\infty du_{M-1} \exp(-zu_{M-1}) \times$$

$$\times \cdots \int_{u_3}^\infty du_2 \exp(-zu_2) \int_{u_2}^\infty du_1 \exp(-zu_1) \ , \tag{A-6}$$

where we have defined

$$F = N(N-1)\cdots(N-M+1) \ , \quad z = 1 - i\xi \ . \tag{A-7}$$

Denote the integral on general term du_m in (A-6 as I_m. Then, it is readily verified that

$$I_1 = \frac{\exp(-zu_2)}{z} \ , \quad I_2 = \frac{\exp(-2zu_3)}{2 \ z^2} \ , \ldots , \quad I_{M-1} = \frac{\exp(-(M-1)zu_M)}{(M-1)! \ z^{M-1}} \ . \tag{A-8}$$

Utilization of the last result in (A-8) yields, from (A-6),

$$f_s(\xi) = \frac{F}{(M-1)! \ z^{M-1}} \int_0^\infty du_M \, [1 - \exp(-u_M)]^{N-M} \exp(-Mzu_M) \ . \tag{A-9}$$

At this point, we use the integral result

$$\int_0^\infty du\ [1 - \exp(-u)]^K \exp(-cu) = \int_0^1 dt\ (1 - t)^K\ t^{c-1} =$$

$$= \frac{\Gamma(c)\ \Gamma(K+1)}{\Gamma(c+K+1)} = \frac{K!}{(c)_{K+1}} \quad \text{for K integer , Re(c) > 0 ,} \qquad \text{(A-10)}$$

which is available from [5; 8.380 1 and 8.384 1]. Then, (A-9) takes on the final form for the characteristic function of the sum s of the M largest random variables of set $\{x_n\}$, namely

$$f_s(\xi) = \frac{F}{(M-1)!\ z^{M-1}} \frac{(N-M)!}{(Mz)_{N-M+1}} = \frac{1}{(1 - i\xi)^{M-1} \prod\limits_{n=M}^{N} \left(1 - i\xi\frac{M}{n}\right)} , \quad \text{(A-11)}$$

where we have used (A-7) to simplify the end result. This compact closed form expression for $f_s(\xi)$ is readily numerically evaluated and is well suited to the numerical methods in [3] for accurately and efficiently determining the false alarm probability of sum random variable s. (More generally, the characteristic function of an arbitrary weighted sum of ordered data $\{x_n'\}$ is accomplished in (B-20) - (B-21)).

It is also interesting to observe that (A-11) corresponds to the characteristic function of a sum of N <u>independent</u> exponential random variables $\{y_n\}$ with <u>non-identical</u> probability density functions

$$p_n(u) = a_n \exp(-a_n u) \quad \text{with} \quad a_n = \begin{Bmatrix} 1 & \text{for} & 1 \le n \le M \\ \frac{n}{M} & \text{for} & M+1 \le n \le N \end{Bmatrix} . \quad \text{(A-12)}$$

The first M random variables of auxiliary set $\{y_n\}$ have mean values 1, while the remainder have decreasing mean values M/n for $M+1 \le n \le N$.

The cumulants of the sum random variable s follow upon expansion of the natural logarithm of (A-11) in a power series in $i\xi$; the k-th cumulant is

$$\chi_s(k) = (k-1)!\left(M + M^k \sum_{n=M+1}^{N} \frac{1}{n^k}\right) \quad \text{for } k \ge 1 . \tag{A-13}$$

In particular, the mean and variance of sum s are

$$\chi_s(1) = M + M \sum_{n=M+1}^{N} \frac{1}{n} , \quad \chi_s(2) = M + M^2 \sum_{n=M+1}^{N} \frac{1}{n^2} . \tag{A-14}$$

The mean of s tends logarithmically to infinity as $N \to \infty$, while the variance and all the other higher-order cumulants of s saturate at finite values as N increases.

The characteristic function $f_s(\xi)$ of sum s in (A-11) can be written in the form

$$f_s(\xi) = \frac{1}{(1 - i\xi)^M} \frac{1}{\prod_{n=M+1}^{N} \left(1 - i\xi\frac{M}{n}\right)} \equiv f_a(\xi)\, f_b(\xi) . \tag{A-15}$$

The probability density functions corresponding to these two factors are expressible in the closed forms

$$p_a(u) = \frac{u^{M-1}\exp(-u)}{(M-1)!} \quad \text{for } u > 0 , \tag{A-16}$$

$$p_b(u) = \frac{N-M}{M} \binom{N}{M} \exp\left(-\frac{M+1}{M}u\right) [1 - \exp(-u/M)]^{N-M-1} \quad \text{for } u > 0 .$$

The latter relation can be verified directly by using result (A-10) with $K = N-M-1$ and $c = M+1-i\xi M$. However, the convolution of the two density functions p_a and p_b in (A-16) has not led to any useful expressions for the probability density function or exceedance distribution function of sum s.

More generally, the following are a Fourier transform pair:

$$p_c(u) = \frac{\beta}{K!} \left(\frac{\alpha}{\beta}\right)_{K+1} \exp(-\alpha u) \left[1 - \exp(-\beta u)\right]^K \quad \text{for } u > 0 \,,$$

$$f_c(\xi) = \left\{\prod_{n=0}^{K}\left(1 - \frac{i\xi}{\alpha + \beta n}\right)\right\}^{-1} ; \quad K \text{ integer} \,, \ \alpha > 0 \,, \ \beta > 0 \,. \quad \text{(A-17)}$$

The k-th cumulant of this pair is

$$\chi_c(k) = (k-1)! \sum_{n=0}^{K} \frac{1}{(\alpha + \beta n)^k} \quad \text{for } k \geq 1 \,, \qquad \text{(A-18)}$$

which is a finite sum of positive terms. This means that the corresponding k-th moment,

$$\mu_c(k) = \int_0^\infty du\, u^k\, p_c(u) = \frac{\beta}{K!} \left(\frac{\alpha}{\beta}\right)_{K+1} \int_0^\infty du\, u^k\, e^{-\alpha u} \left(1 - e^{-\beta u}\right)^K \quad \text{(A-19)}$$

can be expressed as a finite collection of positive terms. Specifically, the k-th moment can be easily and accurately obtained, to high order, by the recursion [6; page 94, (A-6)]

$$\mu_c(k) = \sum_{m=0}^{k-1} \binom{k-1}{m} \chi_c(k-m)\, \mu_c(m) \quad \text{for } k \geq 1 \,, \quad \mu_c(0) = 1 \,, \quad \text{(A-20)}$$

which involves only positive terms. For example,

$$\mu_c(1) = \chi_c(1) \ , \quad \mu_c(2) = \chi_c(2) + \chi_c(1)\ \mu_c(1) = \chi_c(2) + \chi_c^2(1) \ ,$$

$$\mu_c(3) = \chi_c(3) + 2\ \chi_c(2)\ \mu_c(1) + \chi_c(1)\ \mu_c(2) =$$

$$= \chi_c(3) + 3\ \chi_c(2)\ \chi_c(1) + \chi_c^3(1) \ . \qquad \text{(A-21)}$$

By letting t = exp(-u) in (A-19), we have the ability to evaluate the following integral in terms of finite sums of positive quantities:

$$\mu_c(k) = \frac{\beta}{K!} \left(\frac{\alpha}{\beta}\right)_{K+1} \int_0^\infty dt\ [-\ln(t)]^k\ t^{\alpha-1} \left(1 - t^\beta\right)^K \ . \qquad \text{(A-22)}$$

The larger k values will require use of positive recursion (A-20). Parameters k and K are integers, while α and β are positive real.

As an application of the result in (A-17), the characteristic function of the μ-th largest random variable in (A-1), for probability density function (A-5), is immediately found to be

$$f_\mu(\xi) = \left\{\prod_{k=0}^{N-\mu}\left(1 - \frac{i\xi}{\mu + k}\right)\right\}^{-1} = \left\{\prod_{n=\mu}^{N}\left(1 - \frac{i\xi}{n}\right)\right\}^{-1} \ , \qquad \text{(A-23)}$$

which contains only N+1-μ factors. Here, $1 \le \mu \le N$. The k-th cumulant of the μ-th largest random variable is therefore

$$\chi_\mu(k) = (k-1)! \sum_{n=\mu}^{N} \frac{1}{n^k} \quad \text{for } k \ge 1 \ . \qquad \text{(A-24)}$$

In a similar vein, the joint characteristic function of the μ-th largest random variable $\mathbf{x}'_\mu$ and the ν-th largest random variable x'_ν, with $\mu < \nu$, can be found from (A-2) and (A-5) in closed form

$$f_{\mu\nu}(\xi,\zeta) = \overline{\exp(i\xi x'_\mu + i\zeta x'_\nu)} = \iint du\, dv\, g_{\mu\nu}(u,v)\exp(i\xi u + i\zeta v) =$$

$$= \int_{-\infty}^{\infty} dv \exp(i\zeta v) \int_{v}^{\infty} du \exp(i\xi u)\, g_{\mu\nu}(u,v) =$$

$$= \left\{\prod_{n=\mu}^{\nu-1}\left(1 - \frac{i\xi}{n}\right) \prod_{n=\nu}^{N}\left(1 - \frac{i\xi + i\zeta}{n}\right)\right\}^{-1} \quad \text{for } 1 \le \mu < \nu \le N\,. \qquad \text{(A-25)}$$

This form contains a total of $N+1-\mu$ factors, no matter what value the integer ν has. If $\zeta = 0$, (A-25) reduces to (A-23), as expected. On the other hand, if $\xi = 0$, then (A-25) reduces to

$$\left\{\prod_{n=\nu}^{N}\left(1 - \frac{i\zeta}{n}\right)\right\}^{-1}, \qquad \text{(A-26)}$$

which has a form identical to (A-23), again as expected.

The joint cumulants of the μ-th largest random variable x'_μ and the ν-th largest random variable x'_ν of set $\{x_n\}$, with $\mu < \nu$, can be found from the expansion of (A-25) according to

$$\ln f_{\mu\nu}(\xi,\zeta) = \sum_{k=1}^{\infty} \frac{1}{k}\left[(i\xi)^k \sum_{n=\mu}^{\nu-1} \frac{1}{n^k} + (i\xi + i\zeta)^k \sum_{n=\nu}^{N} \frac{1}{n^k}\right]. \qquad \text{(A-27)}$$

Thus, we have

$$\text{mean}\{x'_\mu\} = \sum_{n=\mu}^{N} \frac{1}{n}\,, \quad \text{mean}\{x'_\nu\} = \sum_{n=\nu}^{N} \frac{1}{n}\,, \qquad \text{(A-28)}$$

and

$$\mathrm{var}\{x'_\mu\} = \sum_{n=\mu}^{N} \frac{1}{n^2} , \qquad \mathrm{var}\{x'_\nu\} = \mathrm{cov}\{x'_\mu, x'_\nu\} = \sum_{n=\nu}^{N} \frac{1}{n^2} . \tag{A-29}$$

Observe that the covariance between x'_μ and x'_ν is equal to the variance of x'_ν. Also, the covariance coefficient between x'_μ and x'_ν is

$$\left(\sum_{n=\nu}^{N} \frac{1}{n^2} \Big/ \sum_{n=\mu}^{N} \frac{1}{n^2} \right)^{1/2} ; \quad 1 \le \mu < \nu \le N . \tag{A-30}$$

Thus, for large N, the two largest random variables of a set of N exponential random variables have a covariance coefficient of $(\pi^2-6)^{1/2}/\pi = .626$, while the largest and smallest random variables have a covariance coefficient of $\sqrt{6}/(\pi N) = .780/N$.

More generally, the k,m joint cumulant of x'_μ and x'_ν follows from (A-27) as

$$\chi_{\mu\nu}(k,m) = \begin{Bmatrix} (k-1)!\; S_\mu(k) & \text{for } m = 0 , \; k \ge 1 \\ (k+m-1)!\; S_\nu(k+m) & \text{for } m \ge 1 \end{Bmatrix} , \tag{A-31}$$

where

$$S_j(k) \equiv \sum_{n=j}^{N} \frac{1}{n^k} \quad \text{for } 1 \le j \le N , \quad 1 \le k . \tag{A-32}$$

The characteristic function of the difference between the μ-th largest and the ν-th largest random variables is obtained directly from (A-25) by setting $\zeta = -\xi$:

$$f_{\mu\nu}(\xi,-\xi) = \overline{\exp[i\xi(x'_\mu - x'_\nu)]} = \left\{ \prod_{n=\mu}^{\nu-1} \left(1 - \frac{i\xi}{n}\right) \right\}^{-1} , \tag{A-33}$$

which has $\nu-\mu$ factors. Thus, the difference of adjacent ordered random variables, that is, $\nu = \mu + 1$, has an exponential probability density function with mean value $1/\mu$.

Most generally, it is shown in appendix B that the M-th order joint characteristic function of the μ_1-th largest random variable $\mathbf{x}'_{\mu_1}$, the μ_2-th largest random variable $\mathbf{x}'_{\mu_2}, \ldots,$ and the μ_M-th largest random variable $\mathbf{x}'_{\mu_M}$ of original set $\{\mathbf{x}_n\}$, for $1 \le \mu_1 < \mu_2 < \cdots < \mu_M \le N$, is

$$f(\zeta_1,\zeta_2,\ldots,\zeta_M) = \overline{\exp\left(i\zeta_1\mathbf{x}'_{\mu_1} + i\zeta_2\mathbf{x}'_{\mu_2} + \cdots + i\zeta_M\mathbf{x}'_{\mu_M}\right)} = \tag{A-34}$$

$$= \left\{\prod_{n=\mu_1}^{\mu_2-1}\left(1 - \frac{i\phi_1}{n}\right)\prod_{n=\mu_2}^{\mu_3-1}\left(1 - \frac{i\phi_2}{n}\right) \times \cdots\right.$$

$$\left.\times \prod_{n=\mu_{M-1}}^{\mu_M-1}\left(1 - \frac{i\phi_{M-1}}{n}\right)\prod_{n=\mu_M}^{N}\left(1 - \frac{i\phi_M}{n}\right)\right\}^{-1}, \tag{A-35}$$

where we have defined

$$\phi_m = \zeta_1 + \zeta_2 + \cdots + \zeta_m \quad \text{for } 1 \le m \le M . \tag{A-36}$$

The characteristic function in (A-35) reduces to first-order result (A-11) when we set $\zeta_1 = \zeta_2 = \cdots = \zeta_M = \xi$, and set $\mu_1 = 1$, $\mu_2 = 2,\ldots, \mu_M = M$. A numerical confirmation of general result (A-35) was obtained by simulation for the fourth-order example

$$M = 4,\quad N = 19,\quad \mu_1 = 3,\quad \mu_2 = 6,\quad \mu_3 = 11,\quad \mu_4 = 16,$$
$$\zeta_1 = .31,\quad \zeta_2 = -.53,\quad \zeta_3 = .97,\quad \zeta_4 = .77. \tag{A-37}$$

The exact result from (A-35) is .71239 + i .65022, whereas the simulation yielded estimate .71246 + i .65015, based upon 5,000,000 independent trials of ensemble average (A-34).

An application of (A-35) to the M largest random variables of set $\{\mathbf{x}_n\}$ is afforded by choosing $\mu_m = m$ for $1 \le m \le M$, to obtain the M-th order joint characteristic function

$$f(\zeta_1,\ldots,\zeta_M) = \left\{\prod_{n=1}^{M-1}\left(1 - \frac{i\phi_n}{n}\right) \prod_{n=M}^{N}\left(1 - \frac{i\phi_M}{n}\right)\right\}^{-1} =$$

$$= \left\{\prod_{n=1}^{M}\left(1 - \frac{i\phi_n}{n}\right) \prod_{n=M+1}^{N}\left(1 - \frac{i\phi_M}{n}\right)\right\}^{-1} . \qquad \text{(A-38)}$$

If we set $\zeta_1 = \zeta_2 = \cdots = \zeta_M = \xi$, and use (A-36), this reduces to (A-11).

The results above can be used to find statistics of some nonlinear transformations of ordered data $\{x'_\mu\}$. For example, the sum of the M largest squares is given by

$$\mathbf{z} = \sum_{\mu=1}^{M} \mathbf{x}'^{\,2}_\mu . \qquad \text{(A-39)}$$

From (A-24), we know that

$$\overline{\mathbf{x}'_\mu} = \chi_\mu(1) = \sum_{n=\mu}^{N} \frac{1}{n} , \quad \mathrm{Var}(\mathbf{x}'_\mu) = \chi_\mu(2) = \sum_{n=\mu}^{N} \frac{1}{n^2} . \qquad \text{(A-40)}$$

Therefore, the mean of $\mathbf{z}$ is given by

$$\bar{z} = \sum_{\mu=1}^{M} \left\{ \left(\sum_{n=\mu}^{N} \frac{1}{n} \right)^2 + \sum_{n=\mu}^{N} \frac{1}{n^2} \right\} =$$

$$= \sum_{n,m=1}^{N} \frac{1}{nm} \min(n,m,M) + M \sum_{n=M}^{N} \frac{1}{n^2} + \sum_{n=1}^{M-1} \frac{1}{n} = M\left(\gamma_1^2 + \gamma_2\right) , \quad \text{(A-41)}$$

after considerable manipulation, where

$$\gamma_k \equiv 1 + \sum_{n=M+1}^{N} \frac{1}{n^k} . \quad \text{(A-42)}$$

APPENDIX B. JOINT CHARACTERISTIC FUNCTION OF N ORDERED INDEPENDENT RANDOM VARIABLES

Let $\{x_n\}$ be a set of N independent random variables, where the probability density function of the n-th random variable x_n is p(u,n); the variables need not be identically distributed. Order these random variables into the new set $\{x'_n\}$, where $x'_1 > x'_2 > \cdots > x'_N$. We will evaluate the N-th order joint characteristic function of this ordered set, namely

$$f(\xi_1,\ldots,\xi_N) = \overline{\exp(i\xi_1 x'_1 + \cdots + i\xi_N x'_N)} \; , \qquad \text{(B-1)}$$

for arbitrary values of $\{\xi_n\}$, when probability density functions $\{p(u,n)\}$ of $\{x_n\}$ are exponential. That is, we will consider

$$p(u,n) = \underline{a}(n) \exp[-u \, \underline{a}(n)] \quad \text{for } u > 0 \; , \quad 1 \le n \le N \; . \qquad \text{(B-2)}$$

The ordered random variables $\{x'_n\}$ are highly statistically dependent on each other and are distinctly non-Gaussian.

Observe that knowledge of (B-1) allows ready evaluation of the characteristic function of any weighted sum of the ordered random variables, by simply choosing $\xi_n = \xi \, w_n$. That is, for the weighted sum of the ordered variables,

$$s = \sum_{n=1}^{N} w_n \, x'_n \; , \qquad \text{(B-3)}$$

we have characteristic function

$$f_s(\xi) = \overline{\exp(i\xi s)} = \overline{\exp\left(i\xi \sum_{n=1}^{N} w_n \, x'_n\right)} = f(\xi w_1,\ldots,\xi w_N) \; . \qquad \text{(B-4)}$$

Thus, for example, by choosing only the first M weights of sequence $\{w_n\}$ nonzero, we can investigate the M largest random variables of original set $\{x_n\}$ of size N. Alternatively, for example, by taking only ξ_3 and ξ_5 nonzero in (B-1), we can investigate the combination of the third-largest and fifth-largest random variables in set $\{x_n\}$.

Before we begin the derivation of (B-1), we must consider all possible permutations of the integers 1,2,...,N. For N = 3 for example, there are the N! = 6 possibilities 123, 132, 213, 231, 312, and 321. We label these six sequences according to

$$k_{11} = 1, \quad k_{12} = 2, \quad k_{13} = 3; \qquad k_{21} = 1, \quad k_{22} = 3, \quad k_{23} = 2;$$
$$k_{31} = 2, \quad k_{32} = 1, \quad k_{33} = 3; \qquad k_{41} = 2, \quad k_{42} = 3, \quad k_{43} = 1;$$
$$k_{51} = 3, \quad k_{52} = 1, \quad k_{53} = 2; \qquad k_{61} = 3, \quad k_{62} = 2, \quad k_{63} = 1.$$

In general, the j-th permutation out of the total of N! possible permutations is indicated by the sequence $k_{j1}, k_{j2}, \ldots, k_{jN}$. Thus, we have a matrix of integers $\{k_{jn}\}$ for $1 \le j \le N!$, $1 \le n \le N$.

The joint probability density function of the ordered set of N random variables $\{x'_n\}$ can now be written in the form

$$g(u_1,u_2,\ldots,u_N) = \sum_{j=1}^{N!} \prod_{n=1}^{N} p(u_n,k_{jn}) \quad \text{for } u_1 > u_2 > \cdots > u_N \,, \tag{B-5}$$

and zero otherwise. The N-th order joint characteristic function of the ordered random variables $\{x'_n\}$ is then given by

$$f(\xi_1,\ldots,\xi_N) = \overline{\exp(i\xi_1 x'_1 + \cdots + i\xi_N x'_N)} =$$

$$= \int\cdots\int du_1\cdots du_N \; g(u_1,\ldots,u_N) \exp(i\xi_1 u_1 + \cdots + i\xi_N u_N) =$$

$$= \sum_{j=1}^{N!} \int_{-\infty}^{\infty} du_N \; p(u_N, k_{jN}) \exp(i\xi_N u_N) \int_{u_N}^{\infty} du_{N-1} \; p(u_{N-1}, k_{j,N-1}) \times$$

$$\times \exp(i\xi_{N-1} u_{N-1}) \times \cdots \times \int^{\infty} du_2 \; p(u_2, k_{j2}) \exp(i\xi_2 u_2) \times$$

$$\times \int_{u_2}^{\infty} du_1 \; p(u_1, k_{j1}) \exp(i\xi_1 u_1) \; . \qquad \text{(B-6)}$$

The result in (B-6) is general, holding for any probability density functions $\{p(u,n)\}$. However, one of the few cases where it can actually be evaluated is for the exponential densities given in (B-2). Substitution yields

$$f(\xi_1, \ldots, \xi_N) = \sum_{j=1}^{N!} \int_0^{\infty} du_N \; \underline{a}(k_{jN}) \exp\left(-u_N \; z_{jN}\right) \times$$

$$\times \int_{u_N}^{\infty} du_{N-1} \; \underline{a}(k_{j,N-1}) \; \exp\left(-u_{N-1} \; z_{j,N-1}\right) \times \cdots$$

$$\times \int_{u_3}^{\infty} du_2 \; \underline{a}(k_{j2}) \exp\left(-u_2 \; z_{j2}\right) \int_{u_2}^{\infty} du_1 \; \underline{a}(k_{j1}) \exp\left(-u_1 \; z_{j1}\right) \; , \qquad \text{(B-7)}$$

where we have defined complex quantities

$$z_{jn} = \underline{a}(k_{jn}) - i\xi_n \quad \text{for } 1 \le n \le N \; , \quad 1 \le j \le N! \; . \qquad \text{(B-8)}$$

The factor involving the product of coefficients is given by

$$\underline{a}(k_{jN})\ \underline{a}(k_{j,N-1}) \times \cdots \times \underline{a}(k_{j2})\ \underline{a}(k_{j1}) = \prod_{n=1}^{N} \underline{a}(n) \equiv A\ , \quad \text{(B-9)}$$

regardless of the value of j, the particular permutation.

We denote the integral on u_n in (B-7) by I_{jn}, and let

$$\theta_{jn} = z_{j1} + z_{j2} + \cdots + z_{jn} \quad \text{for } 1 \le n \le N\ , \quad 1 \le j \le N!\ . \quad \text{(B-10)}$$

Then, excluding factor $\underline{a}(k_{jn})$, we find, in order,

$$I_{j1} = \frac{\exp(-u_2\ \theta_{j1})}{\theta_{j1}}\ , \quad I_{j2} = \frac{\exp(-u_3\ \theta_{j2})}{\theta_{j1}\ \theta_{j2}}\ , \ \ldots$$

$$I_{j,N-1} = \frac{\exp(-u_N\ \theta_{j,N-1})}{\theta_{j1} \cdots \theta_{j,N-1}}\ , \quad I_{jN} = \frac{1}{\theta_{j1}\ \theta_{j2} \cdots \theta_{jN}}\ . \quad \text{(B-11)}$$

The use of (B-9) and (B-11) in (B-7) finally yields the joint characteristic function of the ordered random variables $\{x'_n\}$ in the form

$$f(\xi_1,\ldots,\xi_N) = A \sum_{j=1}^{N!} \frac{1}{\theta_{j1}\ \theta_{j2} \cdots \theta_{jN}}\ , \quad \text{(B-12)}$$

where $\{\theta_{jn}\}$ are given by (B-10) and (B-8). By combining these latter expressions, we find, for $1 \le j \le N!$, $1 \le n \le N$,

$$\theta_{jn} = \eta_{jn} - i\psi_n\ , \quad \eta_{jn} = \sum_{p=1}^{n} \underline{a}(k_{jp})\ , \quad \psi_n = \sum_{p=1}^{n} \xi_p\ . \quad \text{(B-13)}$$

The major problem with result (B-12) is the impossibility of evaluating all N! terms, especially for large N. Even if we are interested only in the sum of the first M terms of ordered sequence $\{x'_n\}$, that is, the sum of the M largest random variables of original sequence $\{x_n\}$, that only allows for the simplification of $\{\psi_n\}$ in (B-13) to the form

$$\psi_n = \begin{cases} n\xi & \text{for } \quad 1 \le n \le M \\ M\xi & \text{for } M+1 \le n \le N \end{cases} , \tag{B-14}$$

where we have let $\xi_n = \xi$ for $1 \le n \le M$, and zero otherwise, in (B-1). There is no accompanying simplification in the tedious calculation of $\{\eta_{jn}\}$ in (B-13).

If the exponential probability density functions in (B-2) are characterized by equal signal components in the first $\underline{M}$ terms (without loss of generality) and noise otherwise, then we have the special case

$$\underline{a}(n) = \begin{cases} \underline{a} & \text{for } 1 \le n \le \underline{M} \\ 1 & \text{otherwise} \end{cases} . \tag{B-15}$$

Now, consider the first n terms of the j-th permutation, namely $k_{j1}, k_{j2}, \ldots, k_{jn}$. Let the number of times that any of the numbers 1, 2,..., $\underline{M}$ occurs in these first n locations of permutation j be denoted by L(j,n). Then, from (B-13) and (B-15), we find

$$\eta_{jn} = \underline{a}\, L(j,n) + [n - L(j,n)] = \underline{b}\, L(j,n) + n , \tag{B-16}$$

where we have defined $\underline{b} = \underline{a} - 1$. The joint characteristic function follows from (B-12) as

$$f(\xi_1,\ldots,\xi_N) = \underline{a}^{\underline{M}} \sum_{j=1}^{N!} \prod_{n=1}^{N} \frac{1}{\underline{b}\, L(j,n) + n - i\psi_n} , \tag{B-17}$$

where we also used (B-9) to determine A. This result requires determination of all the integers $\{L(j,n)\}$ for its evaluation, which appears to be a formidable task, in general. Furthermore, this is the simpler case of equal signal components in the first $\underline{M}$ terms.

The special case of no signal in any bins corresponds to $\underline{a} = 1$ in (B-15), thereby giving $\underline{b} = 0$. This causes all the dependence on $\{L(j,n)\}$ in (B-17) to disappear, thereby yielding

$$f(\xi_1,\ldots,\xi_N) = \prod_{n=1}^{N}\left(\frac{1}{1 - i\psi_n/n}\right) , \qquad \text{(B-18)}$$

where sequence $\{\psi_n\}$ is given by (B-13). This is the joint characteristic function (B-1) of ordered sequence $\{x_n'\}$, when all the original random variables $\{x_n\}$ have the common probability density function exp(-u) for u > 0.

As an application of (B-18), consider that all the $\{\xi_n\}$ are zero, except that $\xi_{\mu_1} \neq 0$, $\xi_{\mu_2} \neq 0,\ldots,$ $\xi_{\mu_M} \neq 0$. Then, (B-13) yields

$$\{\psi_n\} = \{0 \cdots 0 \;\; \xi_{\mu_1} \;\; \xi_{\mu_1} \cdots \xi_{\mu_1} \;\; \xi_{\mu_1} + \xi_{\mu_2} \cdots \} , \qquad \text{(B-19)}$$

in which case (B-18) reduces to (A-34) - (A-36).

We can also use (B-18) to determine the characteristic function of the weighted sum s of $\{x_n'\}$ defined in (B-3), for the case of no signal present. Namely, from (B-4) and (B-13) with the identification of ξ_p as $\xi\, w_p$, there follows the characteristic function of the noise-only sum s as

$$f_s(\xi) = \left(\prod_{n=1}^{N}(1 - i\xi W_n)\right)^{-1} , \qquad \text{(B-20)}$$

where the n-th coefficient

$$W_n \equiv \frac{1}{n}\sum_{k=1}^{n} w_k \quad \text{for } 1 \le n \le N \qquad \text{(B-21)}$$

can be interpreted as the average of the first n weights. The k-th cumulant of noise-only sum s follows from (B-20) as

$$\chi_s(k) = (k-1)! \sum_{n=1}^{N} W_n^k \quad \text{for } k \ge 1 . \qquad \text{(B-22)}$$

For the special case of all the weights $\{w_n\}$ equal to 1, then $W_n = 1$ for all n, and (B-20) reduces to $(1 - i\xi)^{-N}$ as expected, since the total sum of the ordered data is equal to the sum of the original data, and there is no signal present. On the other hand, if only the first M weights $\{w_n\}$ are 1, while the remainder are zero, then $W_n = 1$ for $1 \le n \le M$, and $W_n = M/n$ for $M < n \le N$. Then, (B-20) immediately reduces to (A-11), as expected for this noise-only case.

As a partial check on general result (B-17), let $\underline{M} = N$, which means that all N bins contain equal signal components (when signal is present). Then, by the definition under (B-15), we have $L(j,n) = n$, independent of permutation number j. The characteristic function in (B-17) then simplifies to

$$f(\xi_1,\ldots,\xi_N) = \left\{\prod_{n=1}^{N}\left(1 - i\,\frac{\psi_n}{n\underline{a}}\right)\right\}^{-1} , \qquad \text{(B-23)}$$

which is an obvious generalization of (B-18).

As an illustration of the type of analysis required to simplify (B-17) for equal signal components, consider the example of $N = 4$, $\underline{M} = 2$. Of the $N! = 24$ possible sequences that $\{L(j,n)\}$ can take on, there are only 6 different kinds that can occur; they are

1 2 2 2, 1 1 2 2, 1 1 1 2, 0 1 2 2, 0 1 1 2, 0 0 1 2.

Furthermore, each type occurs exactly 4 times. Expression (B-17) then specializes to

$$f(\xi_1,\xi_2,\xi_3,\xi_4) = 4\underline{a}^2\left(\frac{1}{D_1} + \cdots + \frac{1}{D_6}\right) , \tag{B-24}$$

where

$$\begin{aligned}
D_1 &= (\underline{b}+1-i\psi_1)(2\underline{b}+2-i\psi_2)(2\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) , \\
D_2 &= (\underline{b}+1-i\psi_1)(\underline{b}+2-i\psi_2)(2\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) , \\
D_3 &= (\underline{b}+1-i\psi_1)(\underline{b}+2-i\psi_2)(\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) , \\
D_4 &= (1-i\psi_1)(\underline{b}+2-i\psi_2)(2\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) , \\
D_5 &= (1-i\psi_1)(\underline{b}+2-i\psi_2)(\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) , \\
D_6 &= (1-i\psi_1)(2-i\psi_2)(\underline{b}+3-i\psi_3)(2\underline{b}+4-i\psi_4) .
\end{aligned} \tag{B-25}$$

For the numerical example of $\xi_1 = .31$, $\xi_2 = -.53$, $\xi_3 = .97$, $\xi_4 = .77$, $\underline{a} = .71$, the exact answer from (B-24) - (B-25) is .474694 + i .653188, while a simulation result based on 27,000,000 statistically independent trials yielded estimate .474647 + i .653168.

Other numerical examples have indicated that, in general, there are binomial coefficient (N|M) different possible sequences for {L(j,n)}. Although this integer (N|M) can be significantly less than N!, it is still much too large for most practical situations where N is generally much larger than 1. Application of general results (B-12) or (B-17) for the joint characteristic function of the ordered data appears to be limited to very special cases.

APPENDIX C. MAXIMUM DEFLECTION OF WEIGHTED SUM OF ORDERED DATA

The random variable of interest here is

$$z = \sum_{n=1}^{N} w_n \, x'_n \; , \tag{C-1}$$

where original data $\{x_n\}$ is composed of independent and identically-distributed exponential random variables. It has been ordered into descending set $\{x'_n\}$. Under H_0, the characteristic function of z is, from (B-20) and (B-21),

$$f_0(\xi) = \prod_{n=1}^{N}\left(1 - i\xi W_n\right)^{-1} , \quad W_n = \frac{1}{n}\sum_{p=1}^{n} w_p \quad \text{for } 1 \le n \le N \; . \tag{C-2}$$

If we were given coefficients $\{W_n\}$, we can solve for weights $\{w_n\}$ according to (with $W_0 \equiv 0$)

$$w_n = n \, W_n - (n-1) \, W_{n-1} \quad \text{for } 1 \le n \le N \; . \tag{C-3}$$

The mean and variance of z under H_0 are, directly from (C-2),

$$\mu_0 = \sum_{n=1}^{N} W_n \; , \quad \sigma_0^2 = \sum_{n=1}^{N} W_n^2 \; , \quad \chi_0(k) = \sum_{n=1}^{N} W_n^k \; . \tag{C-4}$$

Now, let the means of the n-th random variable x'_n under H_1 and H_0 be μ_{1n} and μ_{0n}, respectively. Define

$$\Delta_n = \mu_{1n} - \mu_{0n} \quad \text{for } 1 \le n \le N \; . \tag{C-5}$$

Then, the difference of the mean outputs of z is, using (C-3),

$$\Delta\bar{z} = \sum_{n=1}^{N} w_n(\mu_{1n} - \mu_{0n}) = \sum_{n=1}^{N} w_n\,\Delta_n =$$

$$= \sum_{n=1}^{N} \Delta_n\left(n\,W_n - (n-1)\,W_{n-1}\right) = \sum_{n=1}^{N} W_n\,n\,(\Delta_n - \Delta_{n+1}) , \qquad \text{(C-6)}$$

where $\Delta_{N+1} \equiv 0$. Also, the deflection of z is

$$d^2 \equiv \frac{(\Delta\bar{z})^2}{\sigma_0^2} = \frac{\left(\sum_{n=1}^{N} W_n\,n\,(\Delta_n - \Delta_{n+1})\right)^2}{\sum_{n=1}^{N} W_n^2} . \qquad \text{(C-7)}$$

The optimum coefficients $\{W_n\}$ for maximum d^2 follow immediately from (C-7) as

$$\tilde{W}_n = \alpha\,n\,(\Delta_n - \Delta_{n+1}) \quad \text{for } 1 \le n \le N , \quad (\alpha \text{ arbitrary}) \qquad \text{(C-8)}$$

giving

$$d^2_{max} = \sum_{n=1}^{N} n^2(\Delta_n - \Delta_{n+1})^2 . \qquad \text{(C-9)}$$

The optimum weights for maximum deflection are, from (C-3) and (C-8) (with $\Delta_0 \equiv 0$),

$$\tilde{w}_n = n\,\tilde{W}_n - (n-1)\,\tilde{W}_{n-1} = \alpha\left(n^2(\Delta_n - \Delta_{n+1}) - (n-1)^2(\Delta_{n-1} - \Delta_n)\right) =$$

$$= \alpha\left(\left[n^2+(n-1)^2\right]\Delta_n - n^2\Delta_{n+1} - (n-1)^2\Delta_{n-1}\right) \quad \text{for } 1 \le n \le N . \qquad \text{(C-10)}$$

Finally, we can scale everything by choice of α so that $\tilde{w}_1 = 1$, without loss of generality; then, $\tilde{W}_1 = 1$.

APPENDIX D. CHARACTERISTIC FUNCTION OF THE SUM OF THE DISTORTED M LARGEST RANDOM VARIABLES OF AN INDEPENDENT SET OF SIZE N

Real random variables $\{x_n\}$, $1 \le n \le N$, are independent and identically distributed, with arbitrary probability density function p, cumulative distribution function C, and exceedance distribution function E. We order this original set of random variables into a new set $\{x'_n\}$, where

$$x'_1 \ge x'_2 \ge \cdots \ge x'_N \ . \tag{D-1}$$

This ordered set of random variables is non-Gaussian, heavily statistically dependent, and not identically distributed.

We select the initial M random variables of the ordered set $\{x'_n\}$, that is, the M largest random variables of original set $\{x_n\}$, and subject them to the common arbitrary memoryless nonlinear transformation h (which could be complex). We then sum these M distorted random variables, obtaining the output random variable s of interest:

$$s = \sum_{n=1}^{M} h(x'_n) \ . \tag{D-2}$$

We are interested in obtaining the exact characteristic function of s, for general N, M, h, and p, despite the deleterious statistical properties of the ordered set $\{x'_n\}$, that were noted under (D-1). In particular, we want the statistical average

$$f_s(\xi) \equiv \overline{\exp(i\xi s)} = \overline{\exp[i\xi h(x'_1) + \cdots + i\xi h(x'_M)]} \ . \tag{D-3}$$

From (A-4), the joint probability density function of the M largest random variables of set $\{x_n\}$ is given by

$$g(u_1,\ldots,u_M) = F\, p(u_1) \cdots p(u_M)\, C(u_M)^{N-M} \tag{D-4}$$

for $u_1 \geq u_2 \geq \cdots \geq u_M$, where $F = N(N-1)\cdots(N+1-M)$. Therefore, average (D-3) can be expressed as the multiple integral

$$f_s(\xi) = F \int_{-\infty}^{\infty} du_M \exp[i\xi h(u_M)]\, p(u_M)\, C(u_M)^{N-M} \times$$

$$\times \int_{u_M}^{\infty} du_{M-1} \exp[i\xi h(u_{M-1})]\, p(u_{M-1}) \times \cdots$$

$$\times \cdots \int_{u_3}^{\infty} du_2 \exp[i\xi h(u_2)]\, p(u_2) \int_{u_2}^{\infty} du_1 \exp[i\xi h(u_1)]\, p(u_1)\ . \tag{D-5}$$

In order to simplify this multiple integral, define function

$$E(u;\xi) \equiv \int_{u}^{\infty} dx \exp[i\xi h(x)]\, p(x)\ . \tag{D-6}$$

This integral is presumed convergent for all u where p is non-zero. Special cases of (D-6) are $E(\infty;\xi) = 0$, and $E(u;0) = E(u)$, which is the ordinary exceedance distribution function of original set $\{x_n\}$. Also, $E(-\infty,\xi)$ is the characteristic function of the output of nonlinear device h subject to a random input with probability density function p. Thus, $E(u;\xi)$ is a mixture of an exceedance distribution and a characteristic function.

Holding parameter ξ fixed, the derivative of $E(u;\xi)$ with respect to u is denoted by a prime, getting

$$E'(u;\xi) = \frac{d}{du} E(u;\xi) = - \exp[i\xi h(u)]\; p(u) \; . \qquad \text{(D-7)}$$

Now, denote the general integral on u_m in (D-5) as I_m. Then, we immediately have $I_1 = E(u_2;\xi)$. Proceeding to the integral on u_2 in (D-5), we can develop it as

$$I_2 = \int_{u_3}^{\infty} du_2 \exp[i\xi h(u_2)]\; p(u_2)\; E(u_2;\xi) =$$

$$= - \int_{u_3}^{\infty} du_2\; E'(u_2;\xi)\; E(u_2;\xi) = \frac{1}{2}\, E(u_3;\xi)^2 \; . \qquad \text{(D-8)}$$

Continuing in this fashion, one integral at a time in (D-5), we arrive at the result for the u_{M-1} integral, namely

$$I_{M-1} = \frac{1}{(M-1)!}\, E(u_M;\xi)^{M-1} \; . \qquad \text{(D-9)}$$

Finally, the last integral on u_M in (D-5) can be expressed as

$$f_s(\xi) = M \binom{N}{M} \int_{-\infty}^{\infty} du \exp[i\xi h(u)]\; p(u)\; C(u)^{N-M}\; E(u;\xi)^{M-1} \; , \qquad \text{(D-10)}$$

where we have simplified the leading constant $F/(M-1)!$. This result in (D-10) holds for $N \geq M$. It is a single integral for the characteristic function of sum random variable s defined in (D-2). If $N > M$, we can integrate by parts on (D-10), using (D-7), to obtain the alternative

$$f_s(\xi) = (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1}\ E(u;\xi)^M . \qquad \text{(D-11)}$$

It should be noted that unequal weights $\{w_n\}$ in sum (D-2) are strictly disallowed in the current analysis. The simplification in (D-8) and (D-9) occurs only when exactly the <u>same</u> h function appears in the successive integrals on u_1 through u_{M-1}. A generalization that includes weights in (D-2) is only possible for very special probability density functions and nonlinear transformations. One such case is exponential p and linear h.

Several checks on these general results are possible. For $\xi = 0$, we have from (D-11) and (D-6),

$$f_s(0) = (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1}\ E(u)^M =$$

$$= (N-M) \binom{N}{M} \int_0^1 dx\ x^{N-M-1}\ (1 - x)^M = 1 , \qquad \text{(D-12)}$$

using [5; 8.380 1 and 8.384 1].

On the other hand, for M = N, we use (D-10), (D-6), and (D-7) to obtain

$$f_s(\xi) = N \int_{-\infty}^{\infty} du\ [-E'(u;\xi)]\ E(u;\xi)^{N-1} = E(-\infty;\xi)^N =$$

$$= \left[\int_{-\infty}^{\infty} dx\ \exp[i\xi h(x)]\ p(x)\right]^N = \overline{\exp[i\xi h(x_n)]}^{\,N} . \qquad \text{(D-13)}$$

This latter expression is recognized as the characteristic function of the sum s, when it is observed that, for M = N,

$$s = \sum_{n=1}^{M=N} h(x_n') = \sum_{n=1}^{N} h(x_n) \quad , \tag{D-14}$$

and that original set $\{x_n\}$ is composed of independent identically distributed random variables with probability density function p.

Finally, for M = 1, (D-10) and (D-7) yield

$$f_s(\xi) = N \int_{-\infty}^{\infty} du \; \exp[i\xi h(u)] \; p(u) \; C(u)^{N-1} =$$

$$= \int_{-\infty}^{\infty} du \; \exp[i\xi h(u)] \; \frac{d}{du} \, C(u)^N = \overline{\exp[i\xi h(x_1')]} \quad . \tag{D-15}$$

The last step in (D-15) is accomplished by observing that the cumulative distribution function of the maximum, x_1', of set $\{x_n\}$ is just $C(u)^N$. And, for M = 1, (D-2) reduces to $s = h(x_1')$.

MEAN OF SUM s IN (D-2)

In order to find the mean of the random variable s defined by (D-2), we begin by expanding (D-6) in a series about $\xi = 0$:

$$E(u;\xi) \sim E(u) + i\xi \; H(u) \quad \text{as } \xi \to 0 \; , \tag{D-16}$$

where

$$H(u) \equiv \int_{u}^{\infty} dx \; h(x) \; p(x) \quad . \tag{D-17}$$

Then, for small ξ, (D-11) yields

$$f_s(\xi) \cong (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1} \left[E(u)^M + i\xi M\ E(u)^{M-1}\ H(u)\right] =$$

$$= 1 + i\xi\ M\ (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1}\ E(u)^{M-1}\ H(u)\ , \qquad \text{(D-18)}$$

upon use of (D-12). Therefore, the mean of sum s is given by the single integral

$$\mu_s = M\ (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1}\ E(u)^{M-1}\ H(u)\ . \qquad \text{(D-19)}$$

A similar expansion of $E(u;\xi)$ and $f_s(\xi)$ to second order gives the second moment of s in the form

$$M\ (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ p(u)\ C(u)^{N-M-1}\ E(u)^{M-2} \left[E(u)\ \underline{H}(u) + (M-1)\ H(u)^2\right] \qquad \text{(D-20)}$$

where

$$\underline{H}(u) \equiv \int_{u}^{\infty} dx\ h(x)^2\ p(x)\ . \qquad \text{(D-21)}$$

EXAMPLE 1: $h(x) = x$, $p(u) = \exp(-u)$ for $u > 0$.

According to (D-2), this corresponds to

$$s = \sum_{n=1}^{M} x_n' \,, \tag{D-22}$$

which is the sum of the M largest exponential random variables. Then, (D-6) yields $E(u;\xi) = \exp(-u(1-i\xi))/(1-i\xi)$ for $u > 0$, and (D-11) becomes, for $N > M$,

$$f_s(\xi) = (N-M) \binom{N}{M} \frac{1}{(1-i\xi)^M} \int_0^\infty du \; e^{-u} \left(1-e^{-u}\right)^{N-M-1} e^{-u(1-i\xi)M} =$$

$$= \binom{N}{M} \frac{N-M}{(1-i\xi)^M} \int_0^1 dx \; (1-x)^{N-M-1} \, x^{(1-i\xi)M} =$$

$$= \left[(1 - i\xi)^M \prod_{n=M+1}^{N} \left(1 - i\xi\frac{M}{n}\right)\right]^{-1} , \tag{D-23}$$

upon letting $x = e^{-u}$, and using [5; 8.380 1 and 8.384 1]. This result agrees with (A-11).

EXAMPLE 2: $h(x) = x^2$, $p(u) = \exp(-u)$ for $u > 0$.

This corresponds to the sum of the squares of the M largest exponential random variables. Now, (D-6) yields [7; chapter 7]

$$E(u;\xi) = \int_u^\infty dx \, \exp(i\xi x^2 - x) = \left(\frac{\pi}{4\xi}\right)^{\frac{1}{2}} \exp\left(i\frac{\pi}{4} + i\xi u^2 - u\right) w(\alpha + i\beta) \tag{D-24}$$

for $u > 0$ and $\xi > 0$, where real quantities

$$\alpha = \left(\frac{\xi}{2}\right)^{\frac{1}{2}} u - \left(\frac{1}{8\xi}\right)^{\frac{1}{2}}, \quad \beta = \left(\frac{\xi}{2}\right)^{\frac{1}{2}} u + \left(\frac{1}{8\xi}\right)^{\frac{1}{2}} . \tag{D-25}$$

It should be observed, that since $\xi > 0$ in (D-24), and $u > 0$ in integral (D-11) for this example, then $\beta > 0$ in (D-25); this means that the real part of w in (D-24) is always positive [7; 7.4.13]. This fact can be used to simplify the calculation of the argument of w, which is needed for (D-11). The final result for the characteristic function of sum s follows from (D-11) in the integral form

$$f_s(\xi) = (N-M) \binom{N}{M} \int_0^\infty du\ e^{-u} \left(1 - e^{-u}\right)^{N-M-1} E(u;\xi)^M , \tag{D-26}$$

which must be done numerically, by means of (D-24).

EXAMPLE 3: $h(x) = x$, $p(u) = \phi(u)$, $C(u) = \Phi(u)$ for all u.

This case corresponds to the sum of the M largest normalized Gaussian random variables; here, $\phi(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2)$ for all u. In this case, (D-6) yields, for all u,

$$E(u;\xi) = \int_u^\infty dx\ \exp(i\xi x)\ \phi(x) = \frac{1}{2} \exp\left(i\xi u - \frac{u^2}{2}\right) w\left(\frac{\xi + iu}{\sqrt{2}}\right) . \tag{D-27}$$

The corresponding characteristic function of sum s is, from (D-11),

$$f_s(\xi) = (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ \phi(u)\ \Phi(u)^{N-M-1}\ E(u;\xi)^M\ . \qquad \text{(D-28)}$$

Extension to a general mean m and standard deviation σ for original set $\{x_n\}$ is easily accomplished; the result is $\exp(i\xi mM)\ f_s(\xi\sigma)$.

EXAMPLE 4: $h(x) = x^2$, $p(u) = \phi(u)$ for all u.

This example corresponds to the sum of the squares of the M largest normalized Gaussian random variables. Now, (D-6) yields, for all u,

$$E(u;\xi) = \int_u^{\infty} dx\ \exp(i\xi x^2)\ \phi(x) =$$

$$= \frac{1}{2(1 - i\xi 2)^{\frac{1}{2}}} \exp\left(- \frac{u^2}{2} (1 - i\xi 2)\right) w\left(u \frac{i}{\sqrt{2}} (1 - i\xi 2)^{\frac{1}{2}}\right)\ . \qquad \text{(D-29)}$$

The characteristic function of sum s is now given by (D-11) as

$$f_s(\xi) = (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ \phi(u)\ \Phi(u)^{N-M-1}\ E(u;\xi)^M\ . \qquad \text{(D-30)}$$

This result is easily extended to a standard deviation $\sigma \neq 1$ for original set $\{x_n\}$, namely $f_s(\xi\sigma^2)$; however, it is not easily extended to mean $m \neq 0$.

The last three examples of characteristic functions rely heavily on an accurate efficient routine for calculating the error function of complex argument, w; see (D-24), (D-27), and (D-29). By contrast, the characteristic function result in (D-23) for the first example is in closed form. The following example will demonstrate a possible limitation of a different kind.

EXAMPLE 5: $h(x) = x$, $p(u) = \exp(-u)\, u^{K-1}/(K-1)!$ for $u > 0$.

As in (D-22), this is the sum of the M largest random variables but where the original random variables $\{x_n\}$ are now chi-squared with 2K degrees of freedom. Then, (D-6) yields closed form

$$E(u;\xi) = \frac{\exp(-u(1-i\xi))}{(1-i\xi)^K} \sum_{k=0}^{K-1} \frac{u^k\,(1-i\xi)^k}{k!} \quad \text{for } u > 0 . \qquad \text{(D-31)}$$

The cumulative distribution function is

$$C(u) = 1 - E(u;0) = 1 - \exp(-u) \sum_{k=0}^{K-1} \frac{u^k}{k!} \quad \text{for } u > 0 . \qquad \text{(D-32)}$$

Substitution of these results in (D-11) yields the characteristic function of sum s in the form

$$f_s(\xi) = (N-M) \binom{N}{M} \int_0^\infty du\; e^{-u} \frac{u^{K-1}}{(K-1)!} \left(1 - e^{-u} \sum_{k=0}^{K-1} \frac{u^k}{k!}\right)^{N-M-1} \times$$

$$\times \left(\frac{\exp(-u(1-i\xi))}{(1-i\xi)^K} \sum_{k=0}^{K-1} \frac{u^k\,(1-i\xi)^k}{k!}\right)^M , \qquad \text{(D-33)}$$

which must be done numerically for each ξ of interest. (For K=1, (D-33) reduces to (D-23).) The latter sum on k in (D-33) is an alternating one, which could be troublesome for large u and/or ξ. A possible alternative is to directly evaluate integral (D-6) numerically; it is a Fourier transform for this example.

MEANS OF SUM s FOR FOUR EXAMPLES

The mean of sum s defined in (D-2) was given generally by integral (D-19), in conjunction with (D-17). For example 1, we find $H(u) = e^{-u}(1+u)$ for $u > 0$, and $\mu_s = M\,\gamma_1$, where

$$\gamma_k \equiv 1 + \sum_{n=M+1}^{N} \frac{1}{n^k} \quad \text{for } k \ge 1 \; . \tag{D-34}$$

More generally, the k-th cumulant of sum s for this example, $\chi_s(k)$, is given in (A-13).

For example 2, there follows $H(u) = 2\,e^{-u}(1 + u + u^2/2)$ for $u > 0$, and $\mu_s = M(\gamma_1^2 + \gamma_2)$, where we used (A-17) - (A-21). The results for mean μ_s, in both of these examples, involve only sums of positive terms; see (D-34).

For example 3, we have

$$H(u) = \int_u^\infty dx\; x\;\phi(x) = \phi(u) \quad \text{for all } u \; , \tag{D-35}$$

thereby giving

$$\mu_s = M\ (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ \phi(u)^2\ \Phi(u)^{N-M-1}\ [1 - \Phi(u)]^{M-1}\ , \tag{D-36}$$

which must be done numerically.

For example 4, $H(u) = u\ \phi(u) + 1 - \Phi(u)$, yielding

$$\mu_s = M + M\ (N-M) \binom{N}{M} \int_{-\infty}^{\infty} du\ u\ \phi(u)^2\ \Phi(u)^{N-M-1}\ [1 - \Phi(u)]^{M-1}. \tag{D-37}$$

Finally, for example 5, we have

$$H(u) = K\ \exp(-u) \sum_{k=0}^{K} \frac{u^k}{k!} \quad \text{for } u > 0\ , \tag{D-38}$$

with mean

$$\mu_s = \frac{K\ M\ (N-M)}{(K-1)!} \binom{N}{M} \int_{0}^{\infty} du\ \exp(-u(M+1)) \left(\sum_{k=0}^{K} \frac{u^{k+K-1}}{k!}\right) \times$$

$$\times \left(1 - e^{-u} \sum_{k=0}^{K-1} \frac{u^k}{k!}\right)^{N-M-1} \left(\sum_{k=0}^{K-1} \frac{u^k}{k!}\right)^{M-1} . \tag{D-39}$$

APPENDIX E. RECEIVER OPERATING CHARACTERISTICS FOR SUM-OF-M-LARGEST PROCESSOR

The sum-of-M-largest (SOML) processor is characterized according to (28) by the summation

$$z = \sum_{n=1}^{M} x_n' \gtrless v \ . \tag{E-1}$$

Here, random variables $\{x_n'\}$ are the ordered version of given data $\{x_n\}$ for $1 \le n \le N$. Summation limit M is a hypothesized or assumed value for the number of bins occupied by signal. The actual number of occupied bins is $\underline{M}$. The following receiver operating characteristics (ROC) are plotted for N = 1024, and for $\underline{M}$ and M taking on the values listed in (34) and (35), respectively. The quantity $\underline{S}$(dB) is the common value of the signal power per bin in dB. Since the noise power per bin has been normalized at unity, $\underline{S}$ is also the signal-to-noise ratio per bin. The number of trials utilized for the detection probability in each case is noted on the individual figure. The false alarm probability was determined exactly from characteristic function (29). The bottom-most straight line at 45° corresponds to $\underline{S} = 0$, that is, $\underline{S}(\text{dB}) = -\infty$ dB; it lies along $P_d = P_f$.

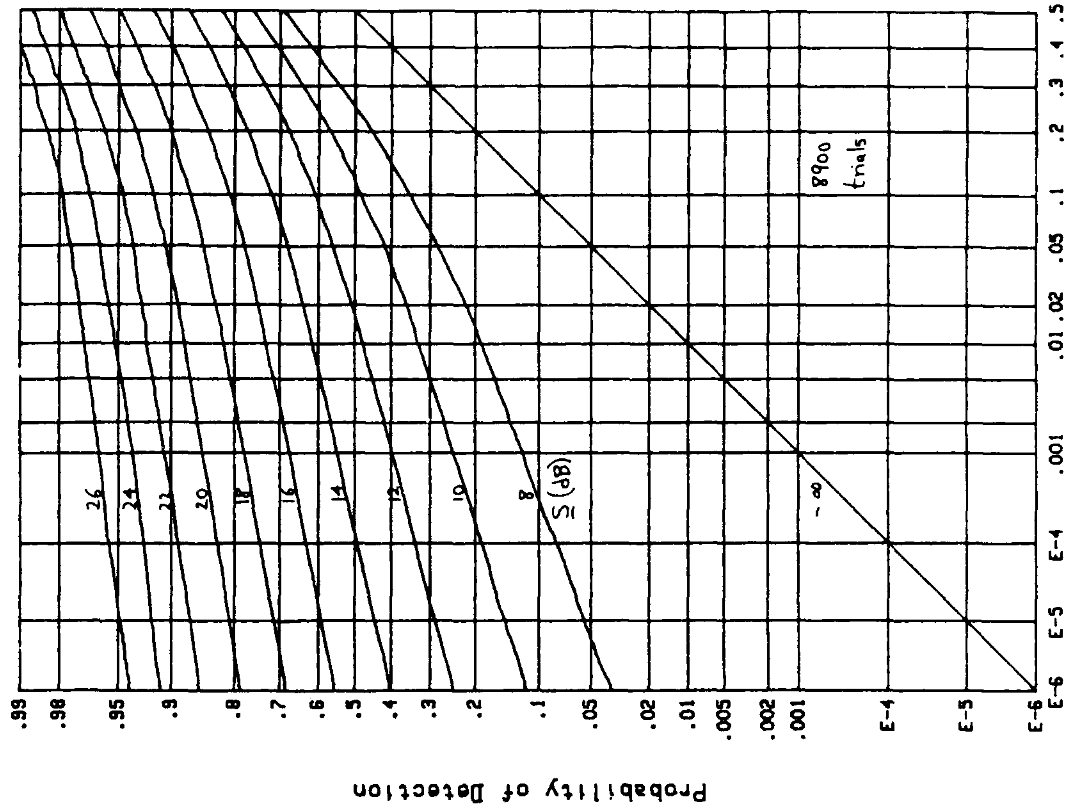

Figure E-2. ROC for SOML, M=1, M=3

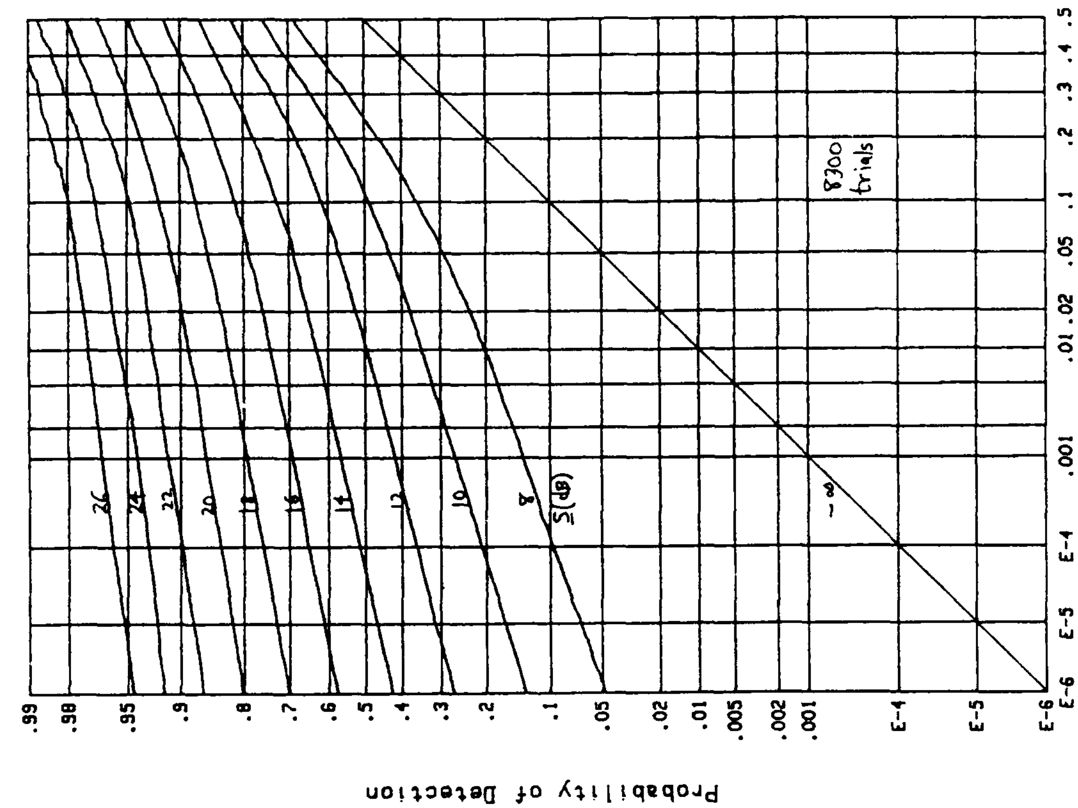

Figure E-1. ROC for SOML, M=1, M=2

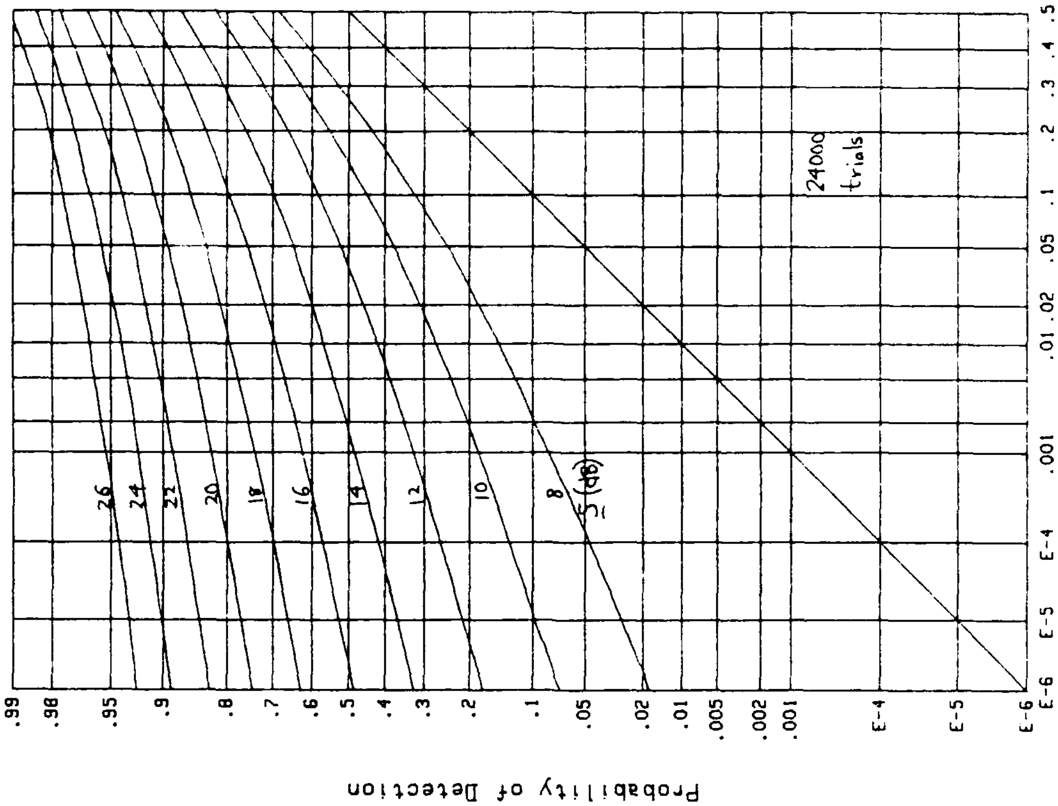

Figure E-4. ROC for SOML, M=1, M=8

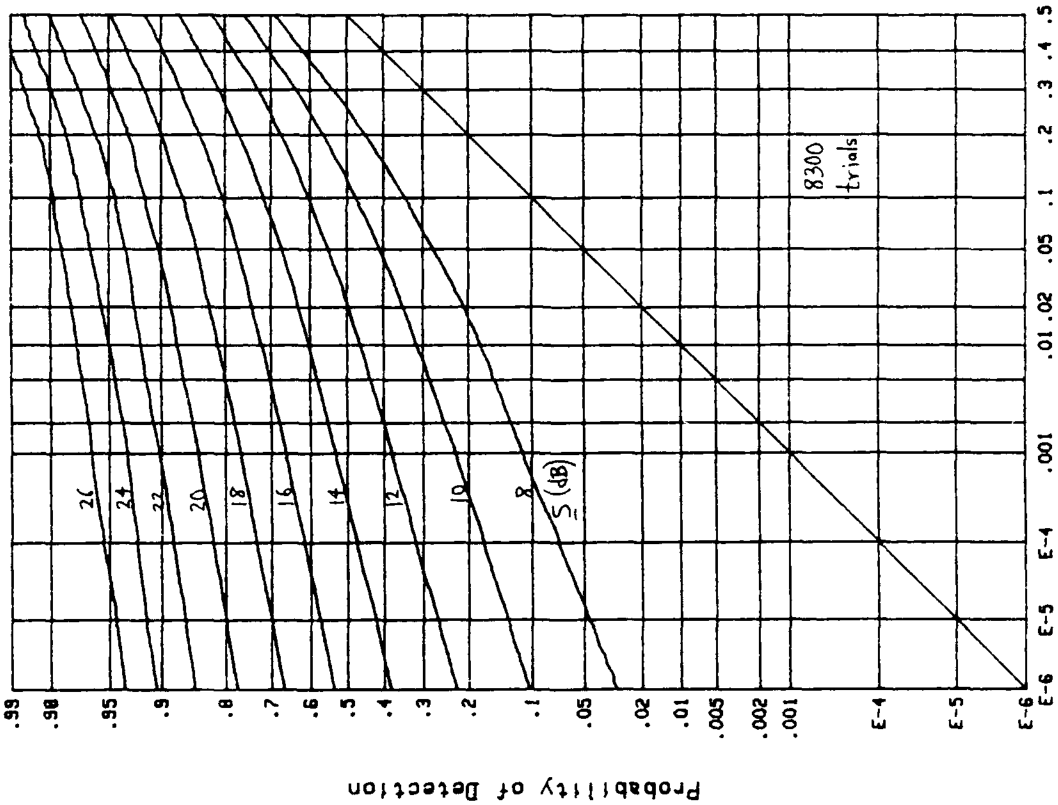

Figure E-3. ROC for SOML, M=1, M=4

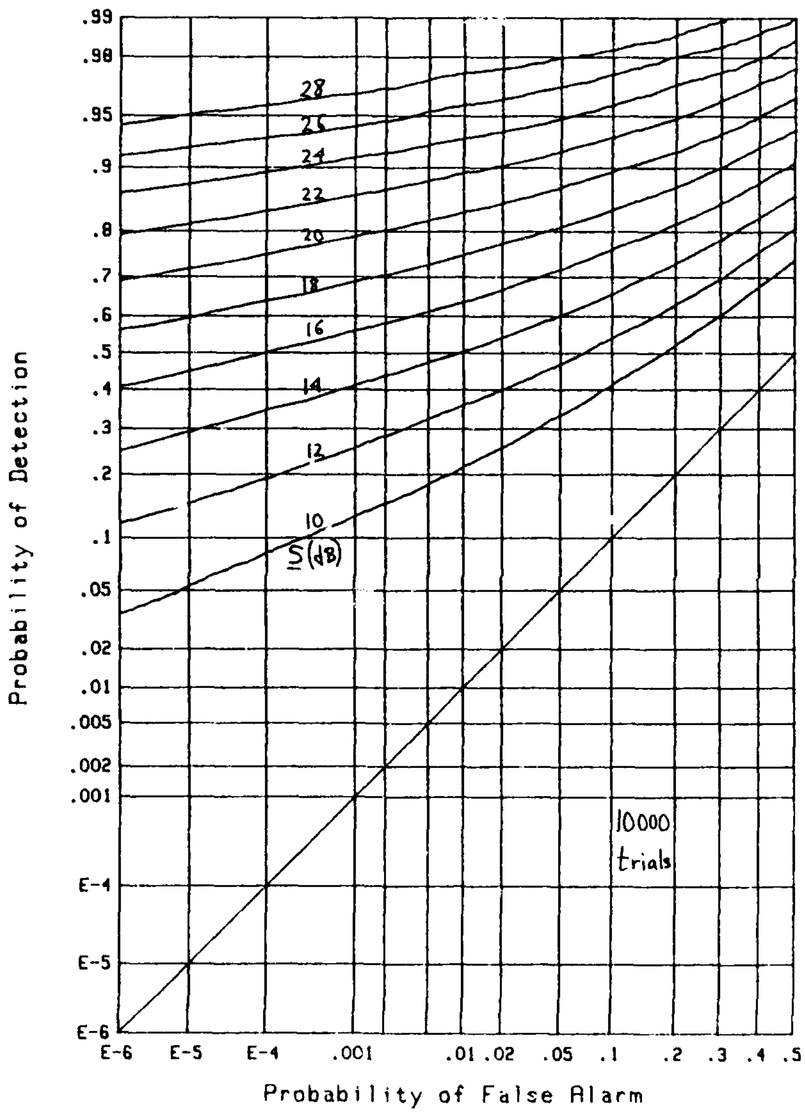

Figure E-5. ROC for SOML, M=1, M=16

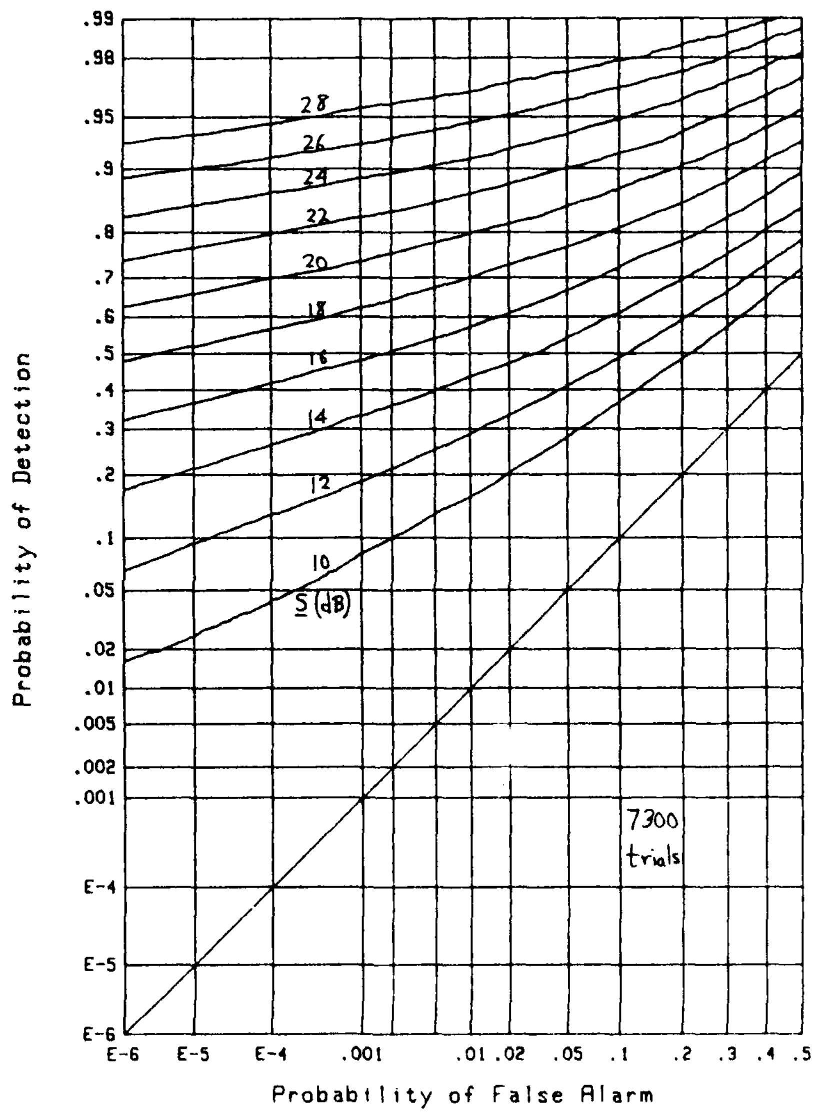

Figure E-6. ROC for SOML, M=1, M=32

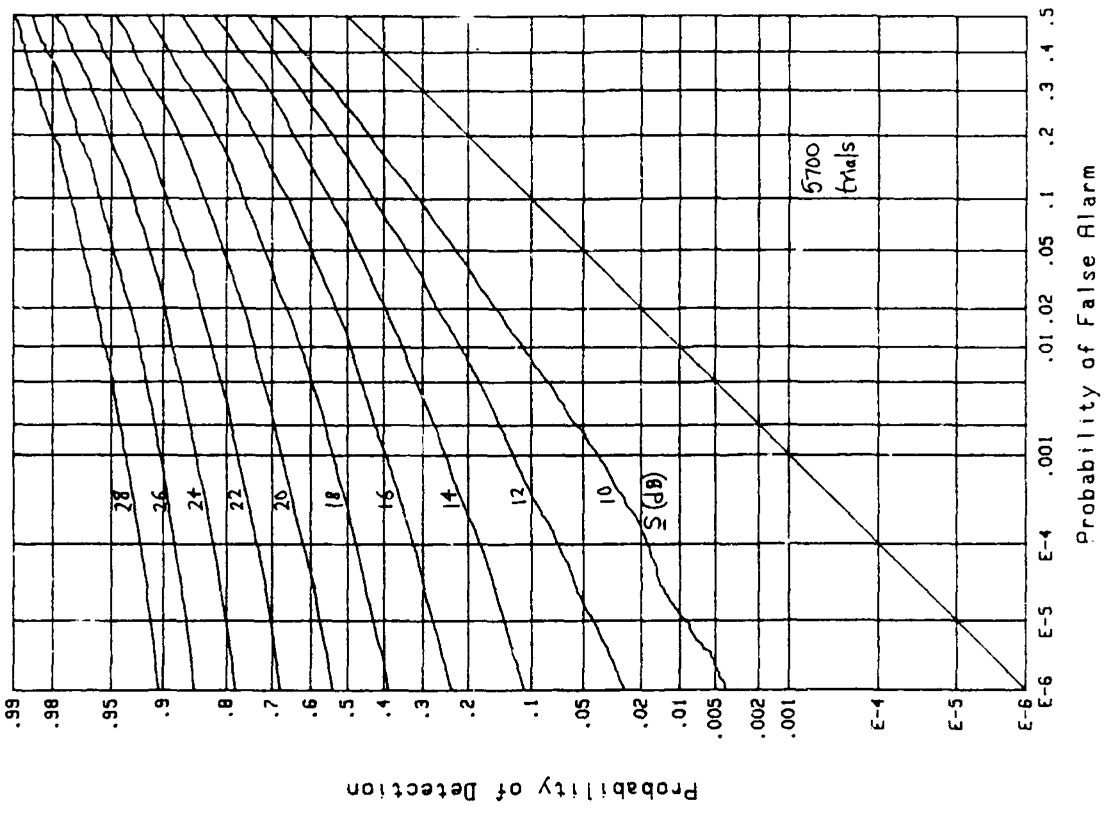

Figure E-7. ROC for SOML, $\underline{M}$=1, M=64

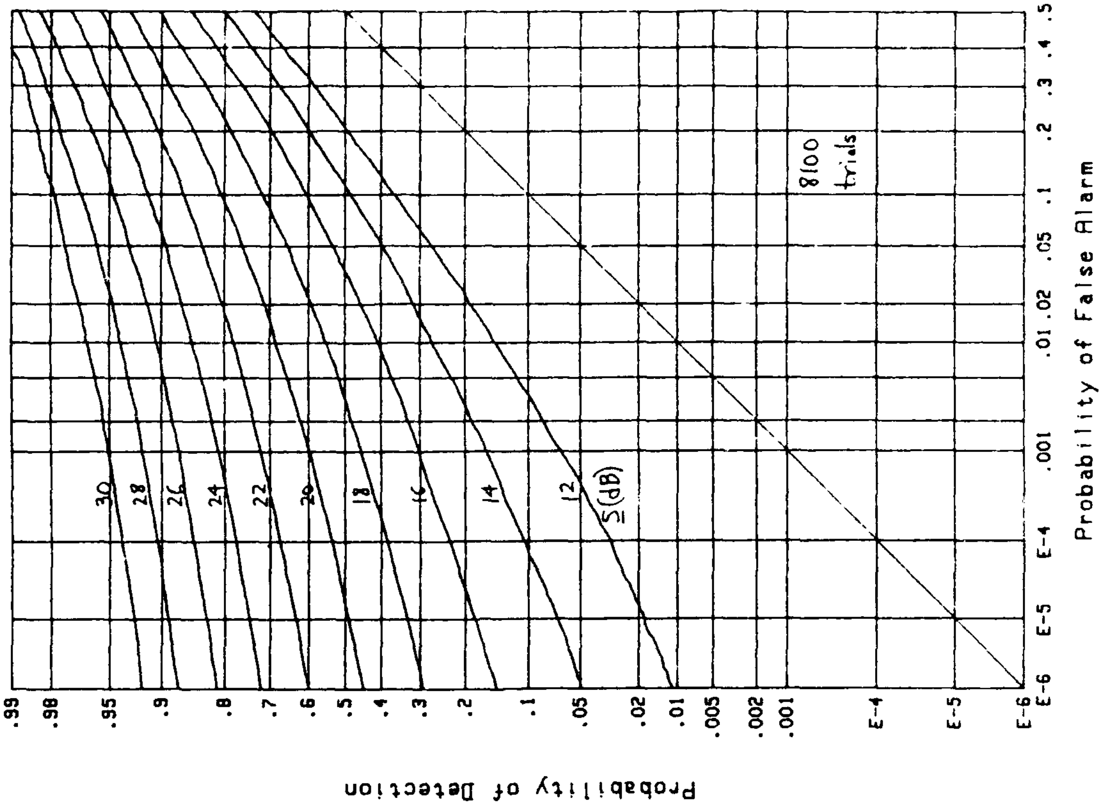

Figure E-8. ROC for SOML, $\underline{M}$=1, M=128

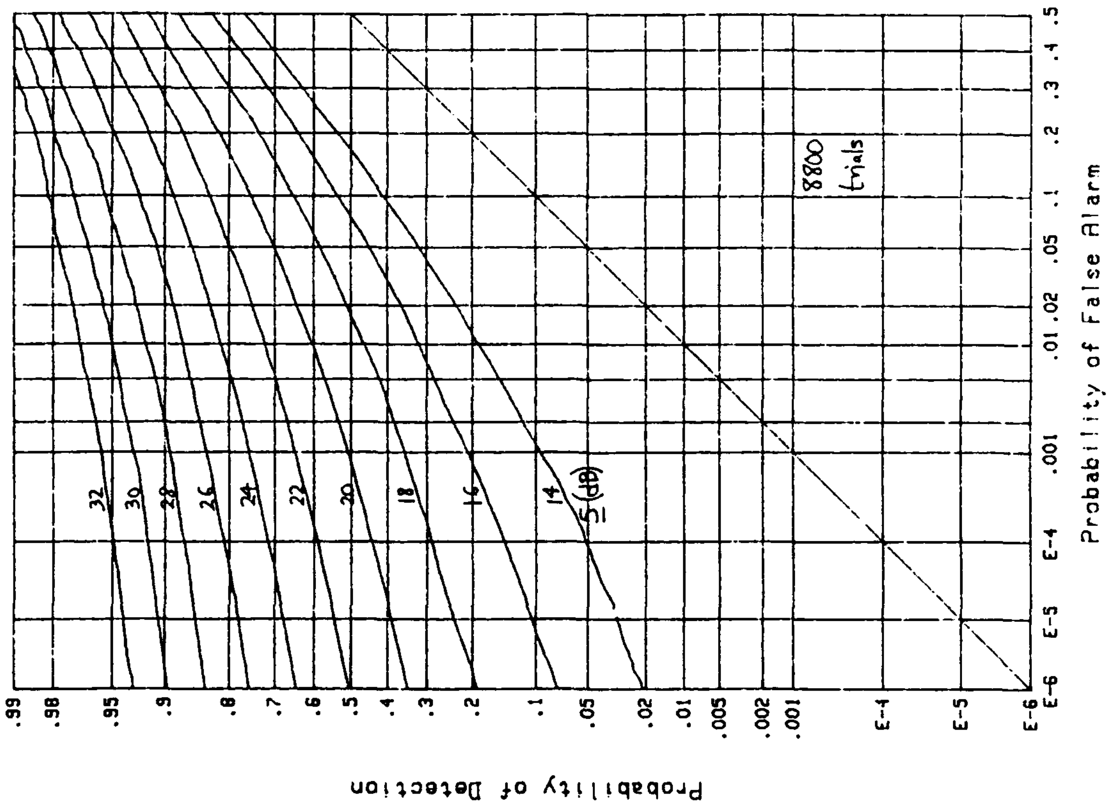

Figure E-9. ROC for SOML, M=1, M=256

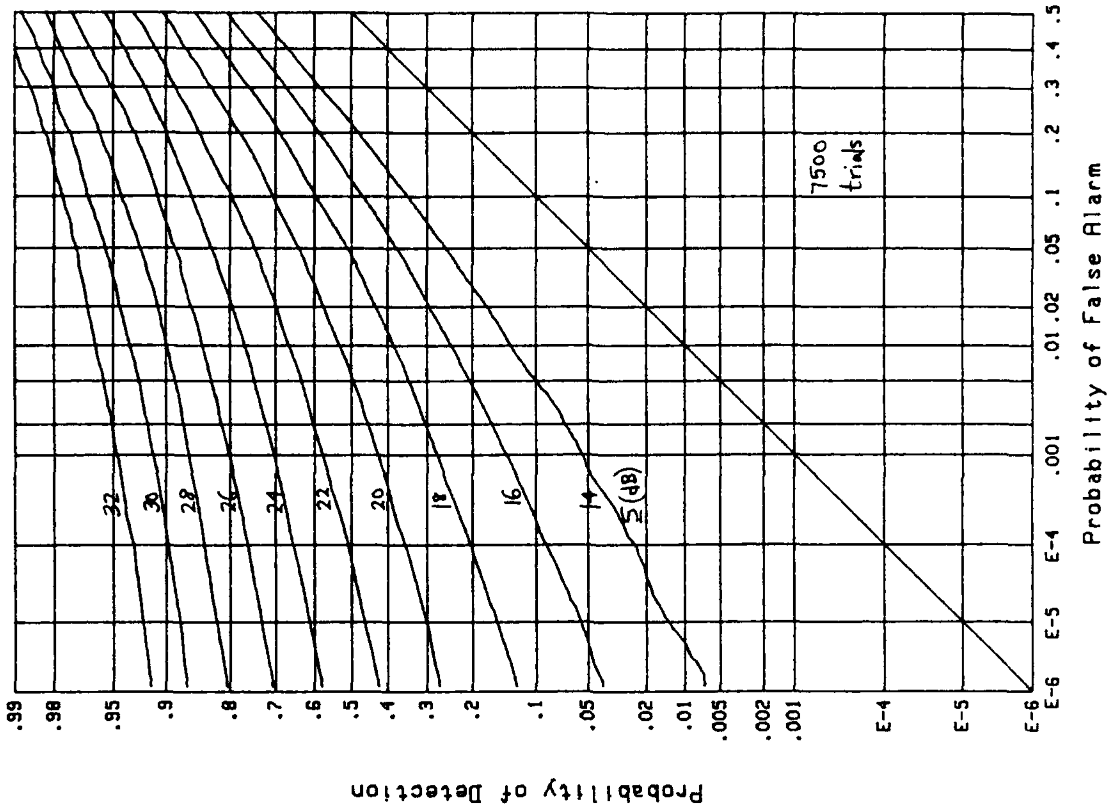

Figure E-10. ROC for SOML, M=1, M=512

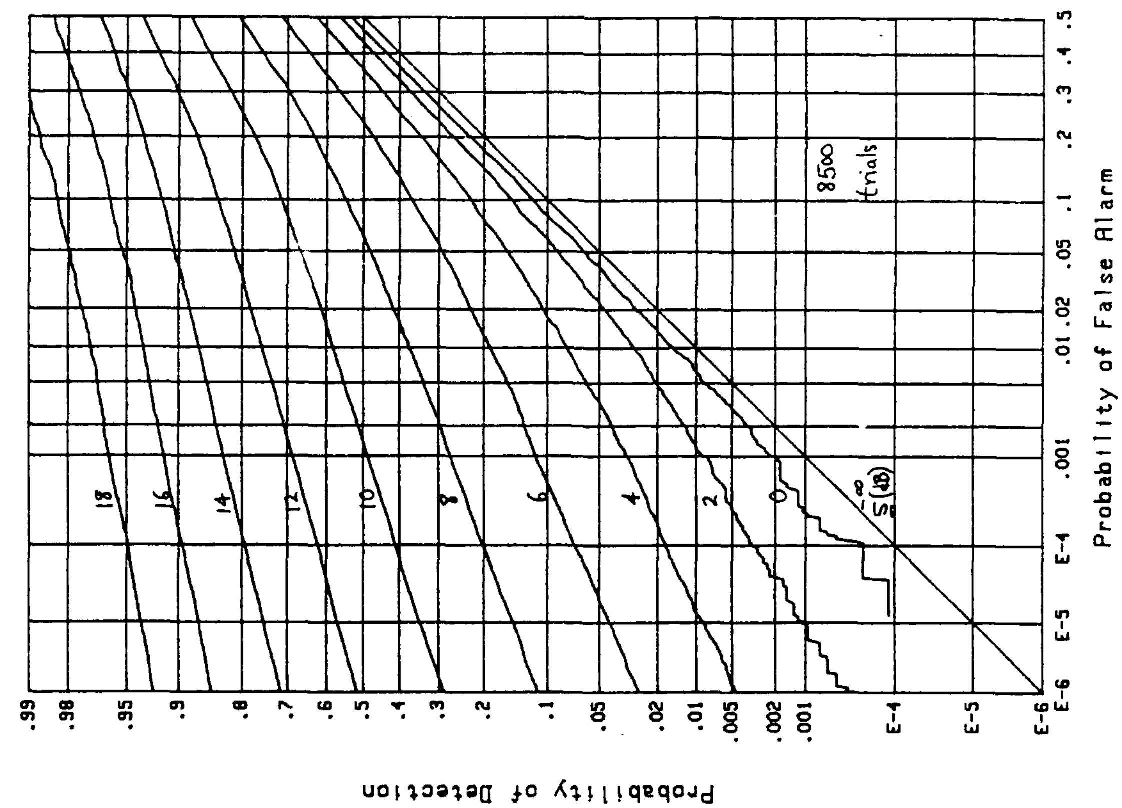

Figure E-11. ROC for SOML, $\underline{M}$=2, M=2

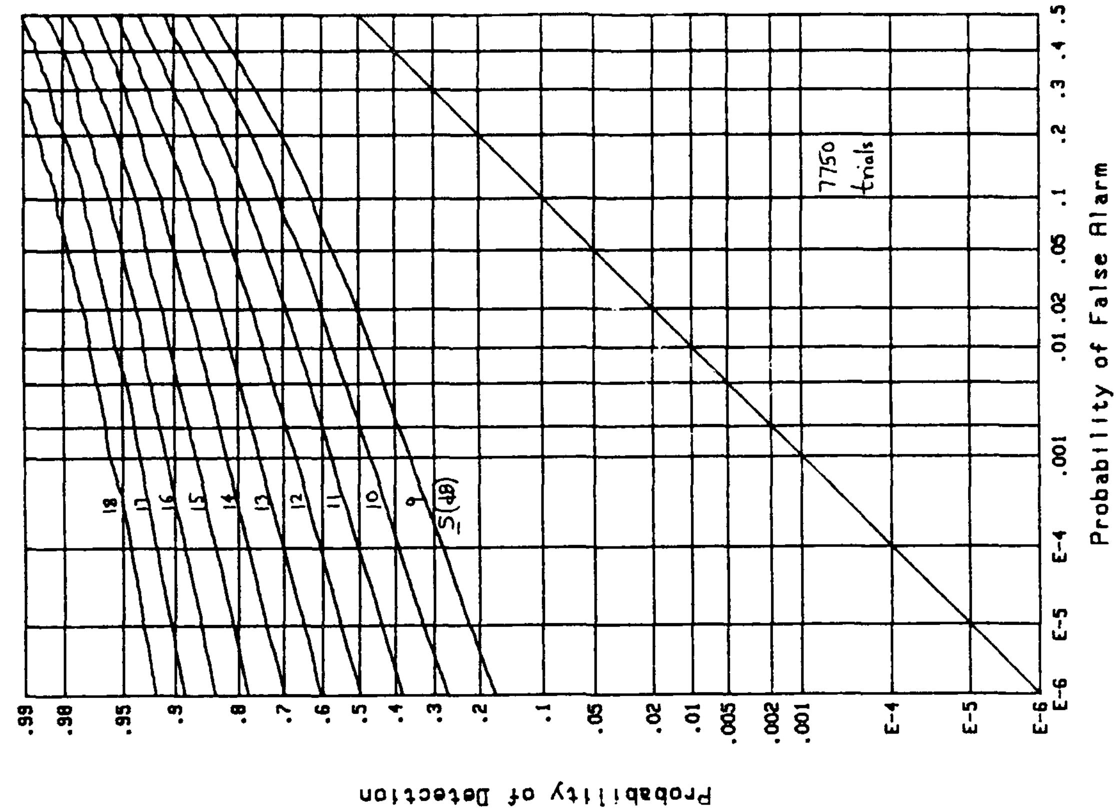

Figure E-12. ROC for SOML, $\underline{M}$=2, M=3

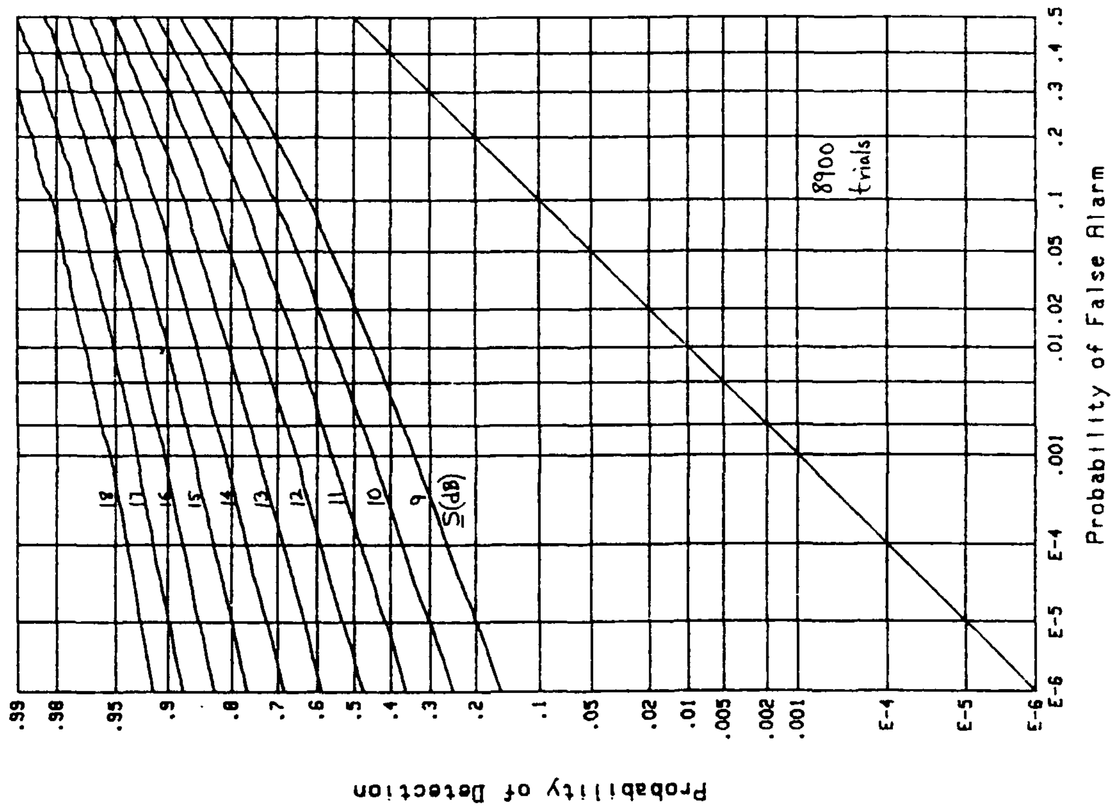

Figure E-13. ROC for SOML, M=2, M=4

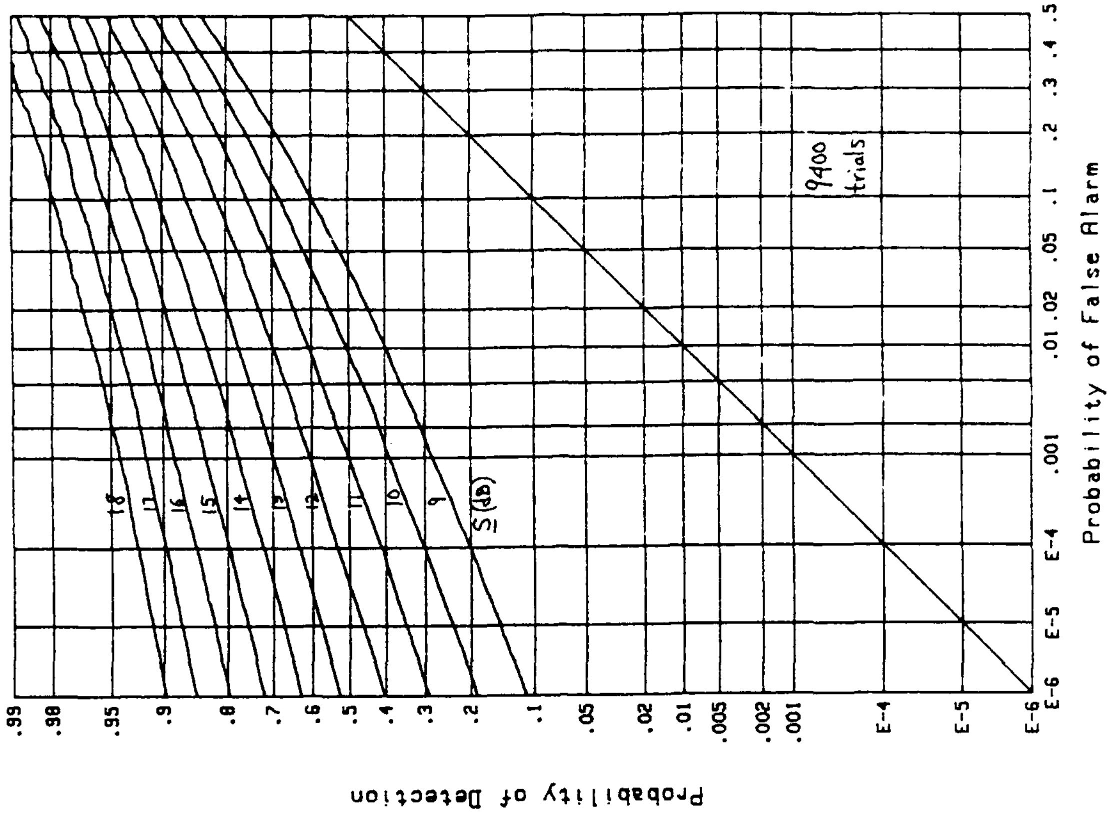

Figure E-14. ROC for SOML, M=2, M=8

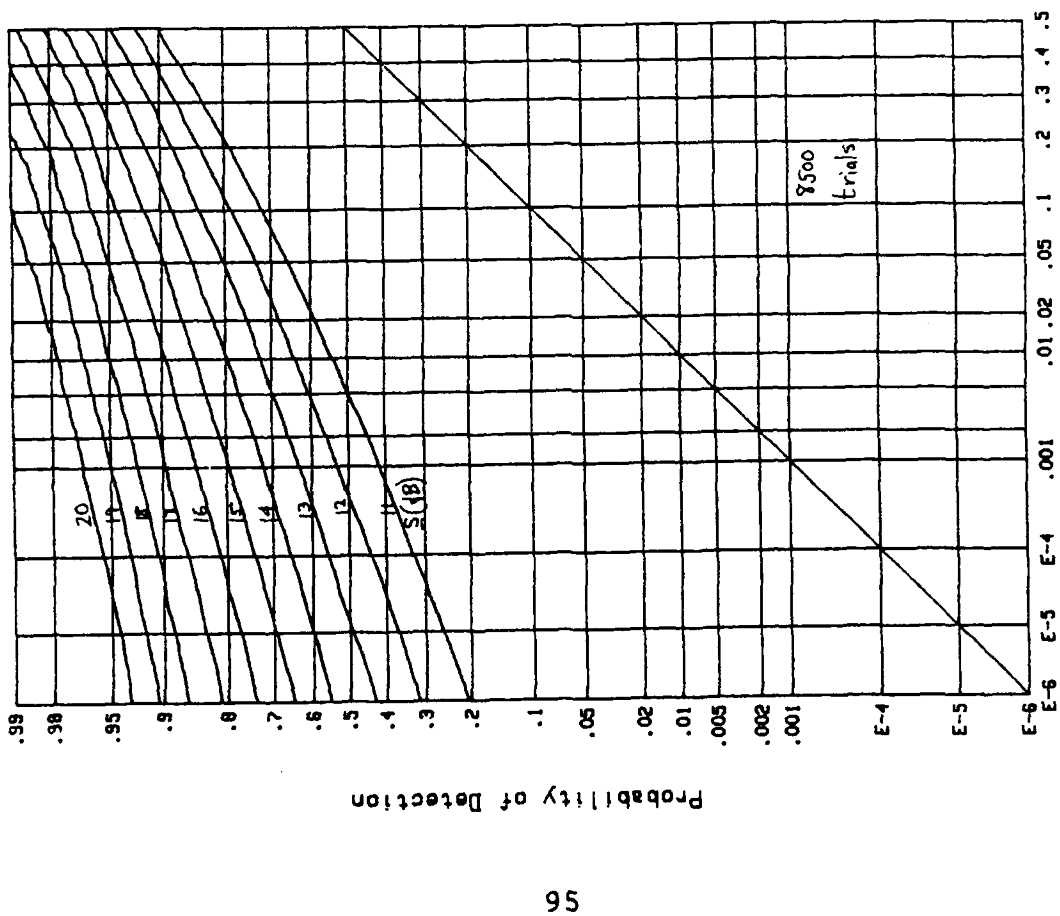

Figure E-15. ROC for SOML, $\underline{M}=2$, M=16

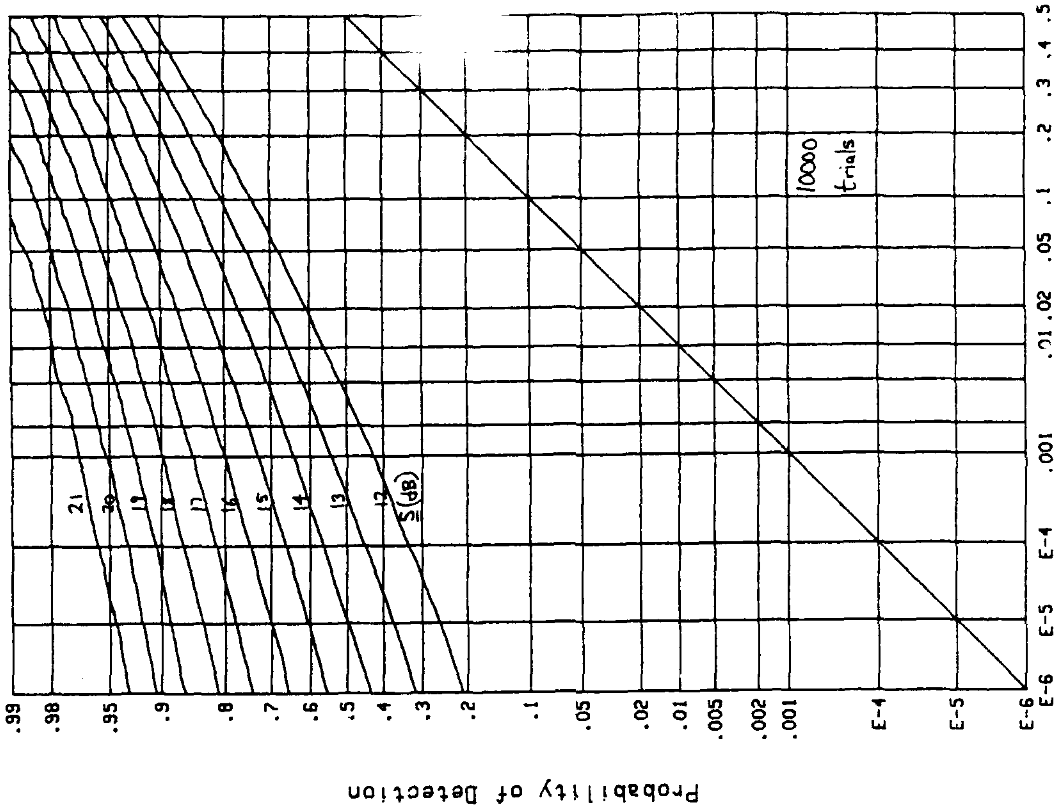

Figure E-16. ROC for SOML, $\underline{M}=2$, M=32

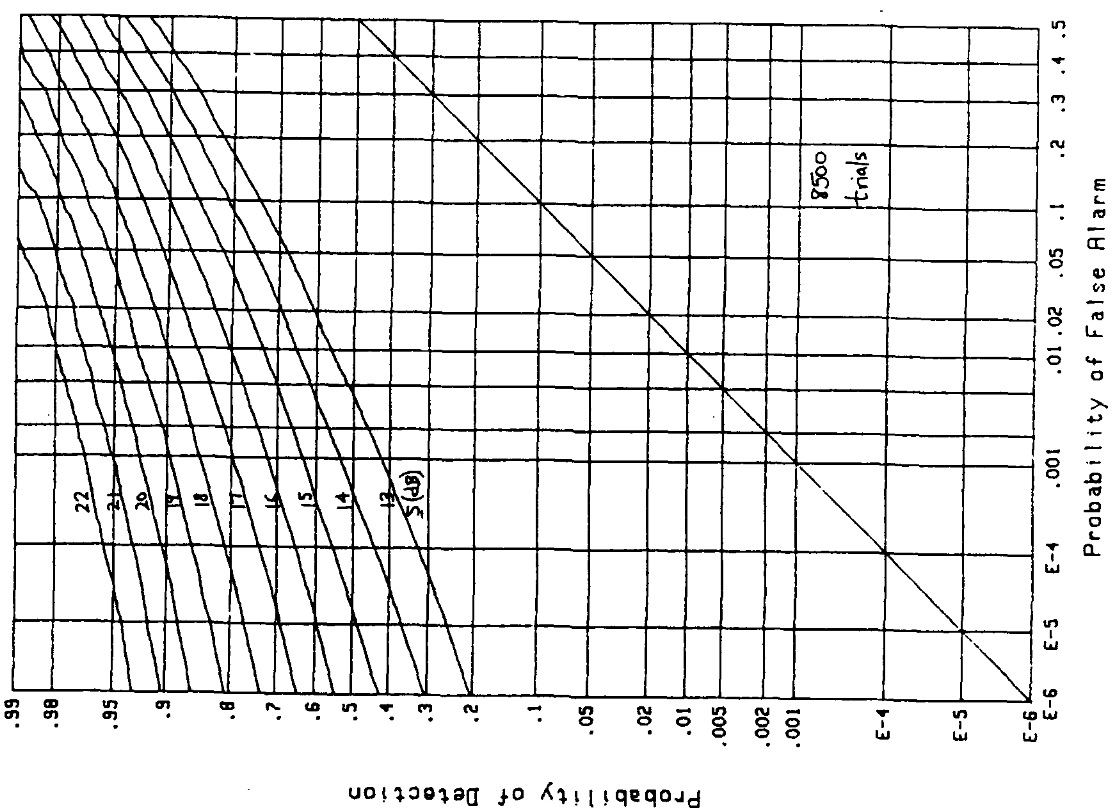

Figure E-17. ROC for SOML, M=2, M=64

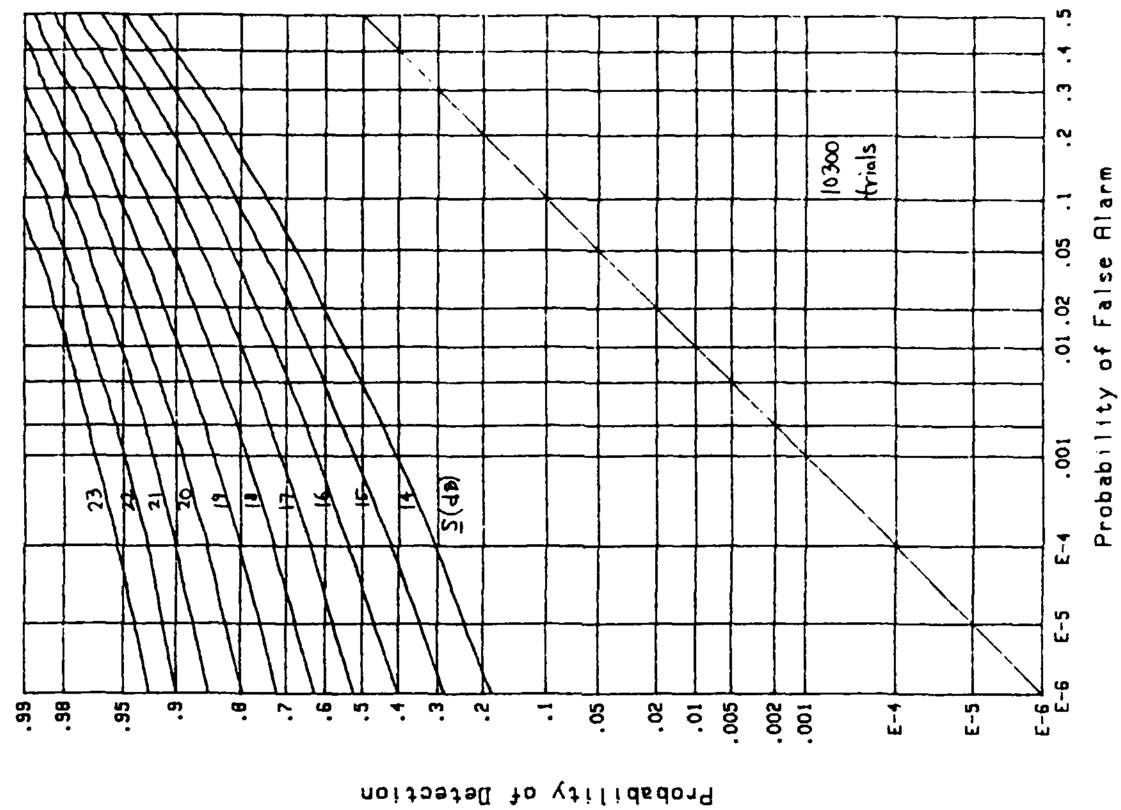

Figure E-18. ROC for SOML, M=2, M=128

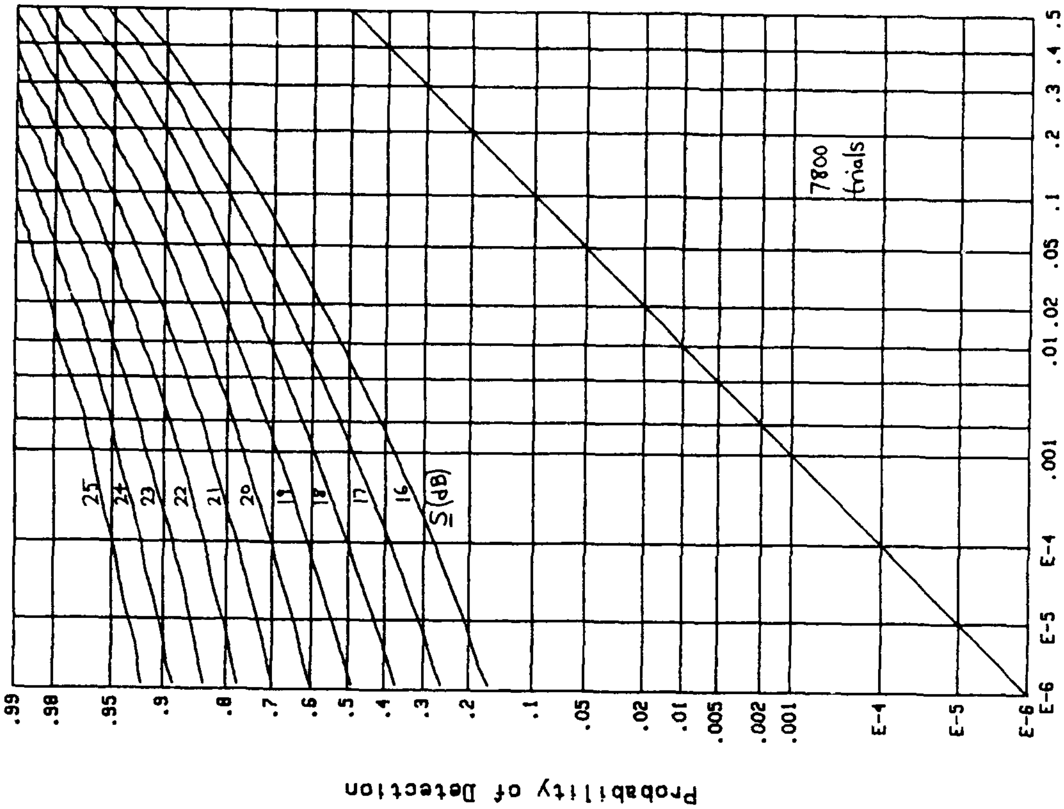

Figure E-20. ROC for SOML, M=2, M=512

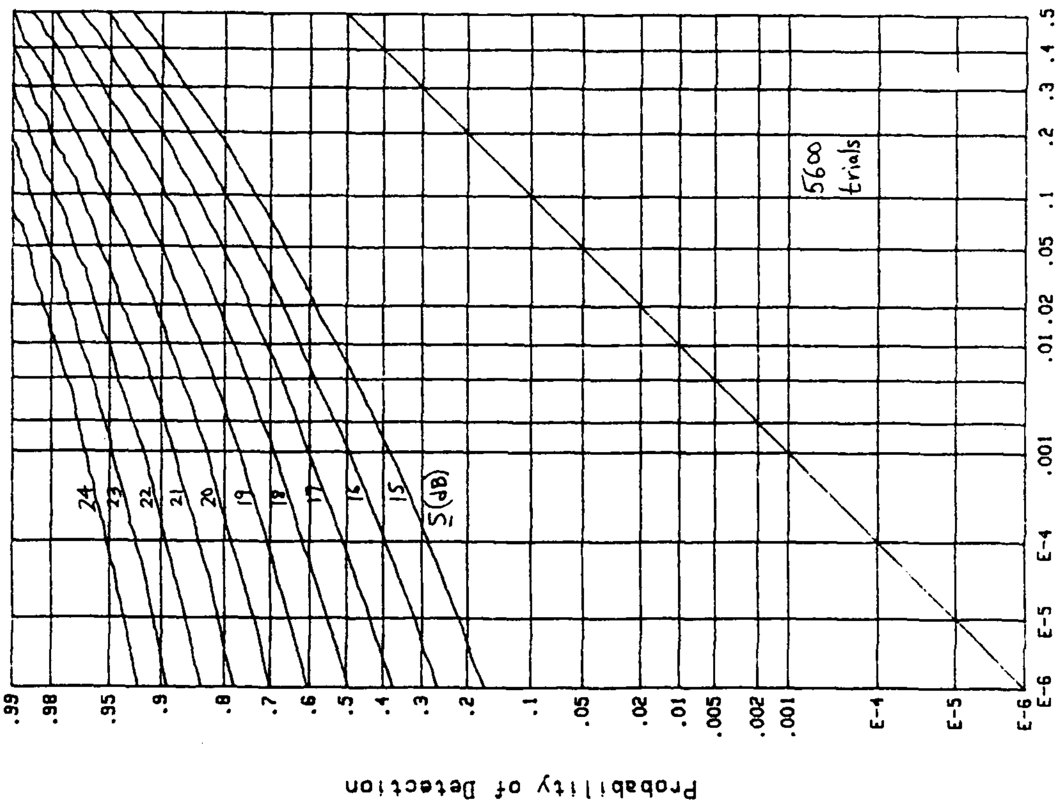

Figure E-19. ROC for SOML, M=2, M=256

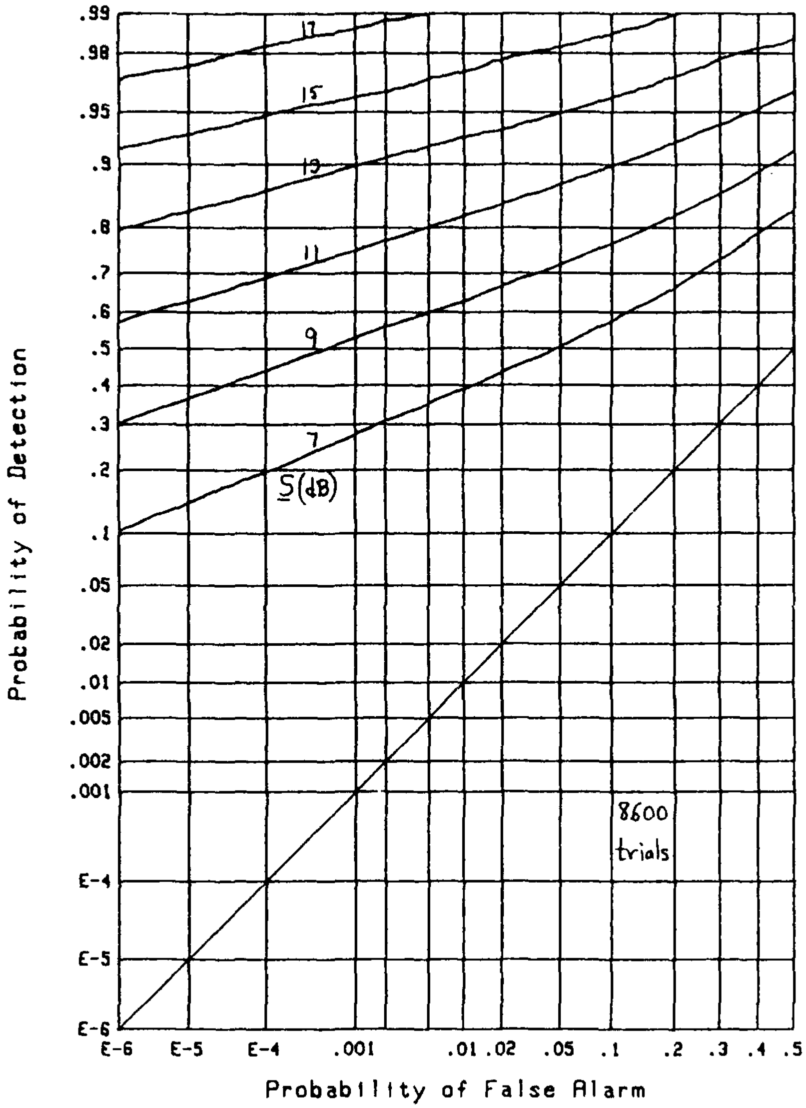

Figure E-21. ROC for SOML, M=3, M=2

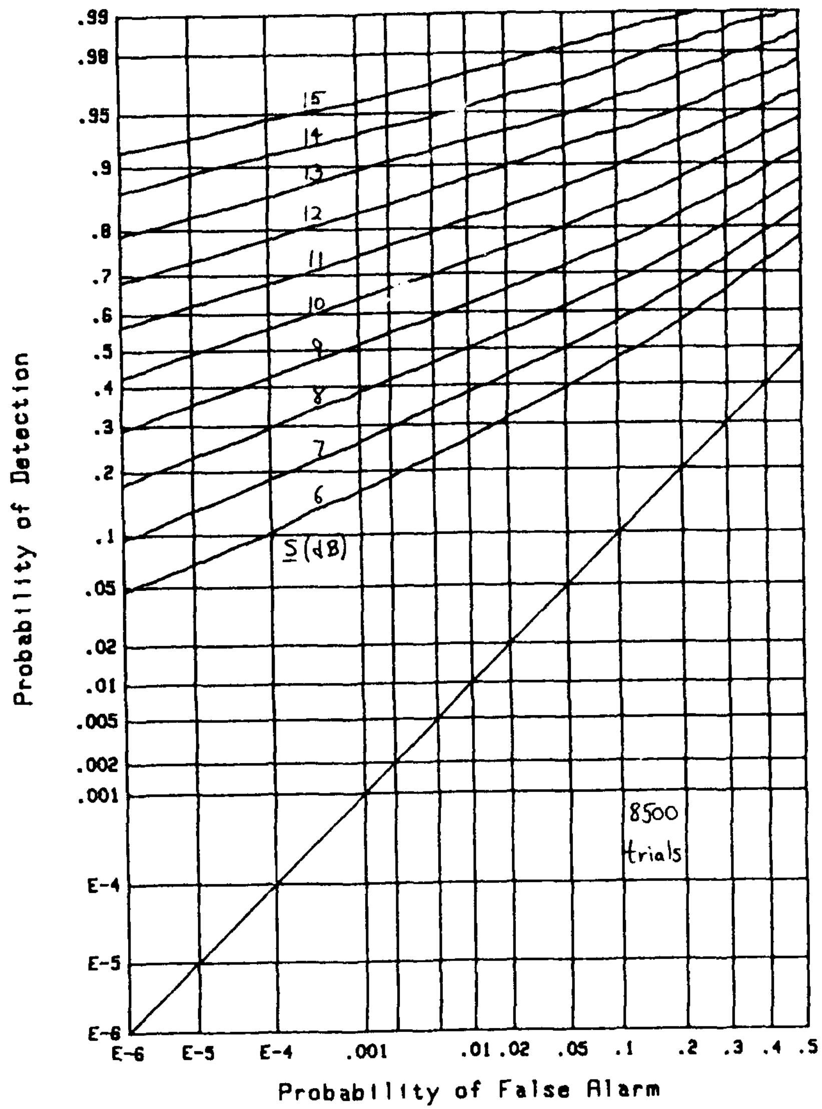

Figure E-22. ROC for SOML, M=3, M=3

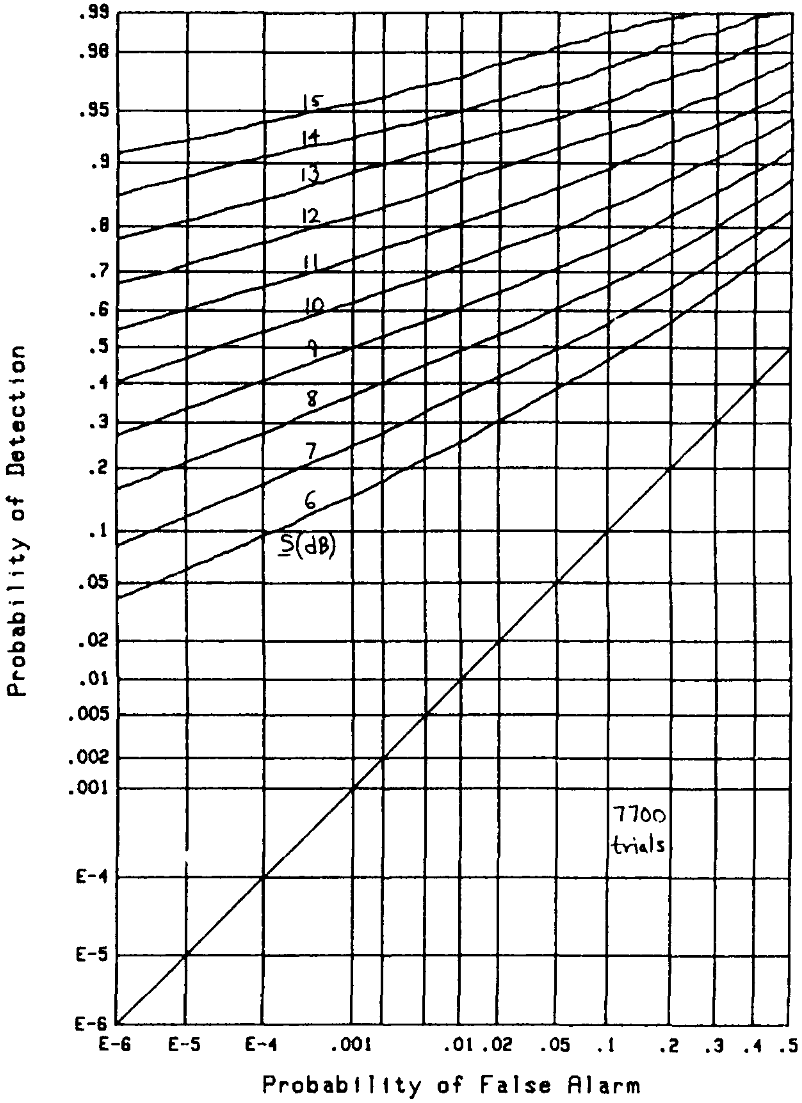

Figure E-23. ROC for SOML, M=3, M=4

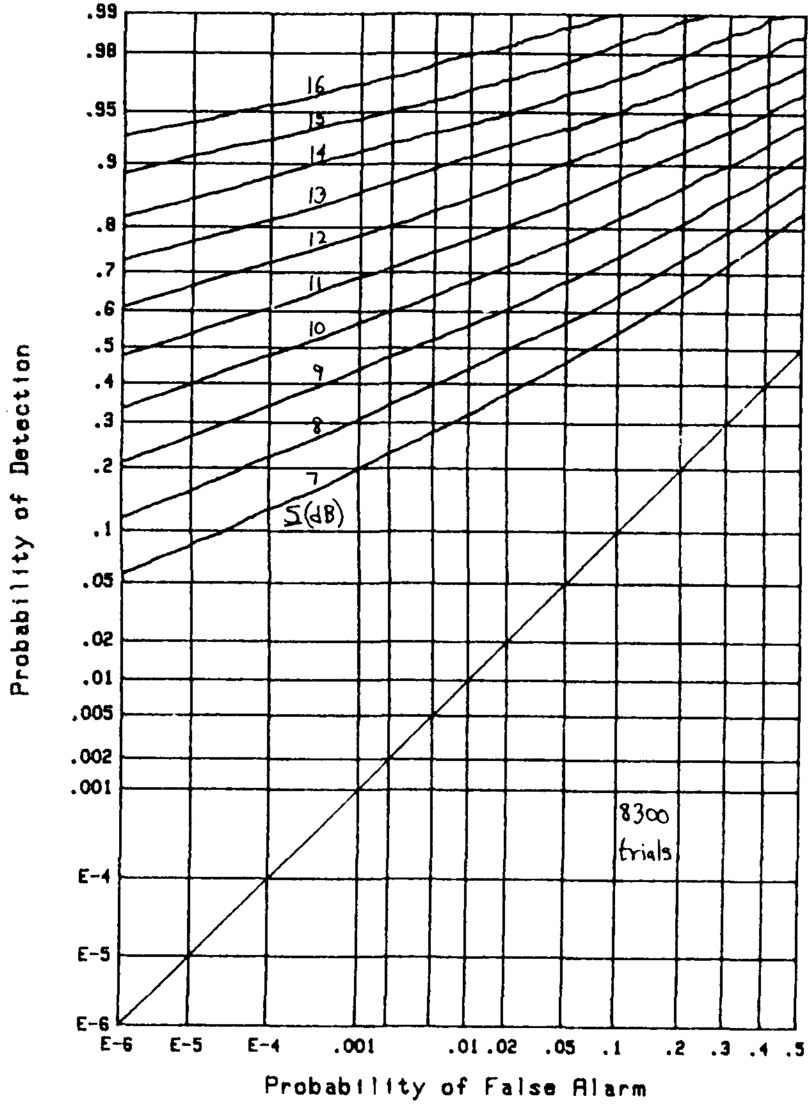

Figure E-24. ROC for SOML, M=3, M=8

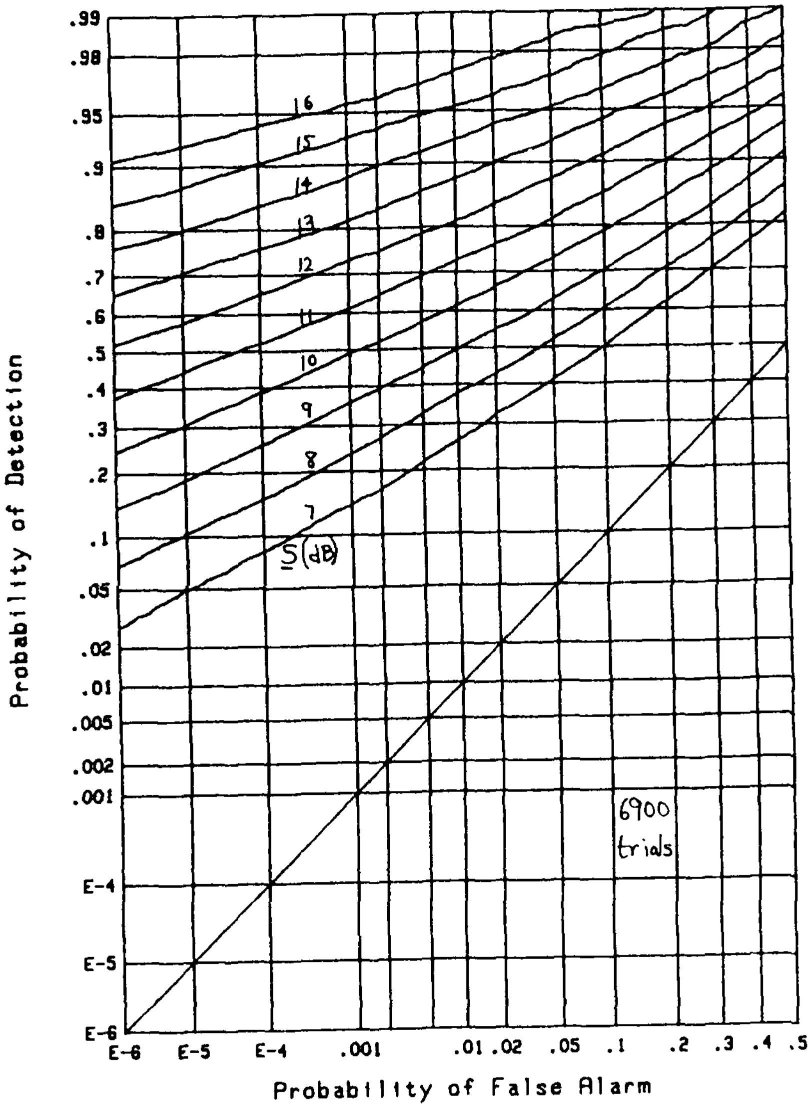

Figure E-25. ROC for SOML, M=3, M=16

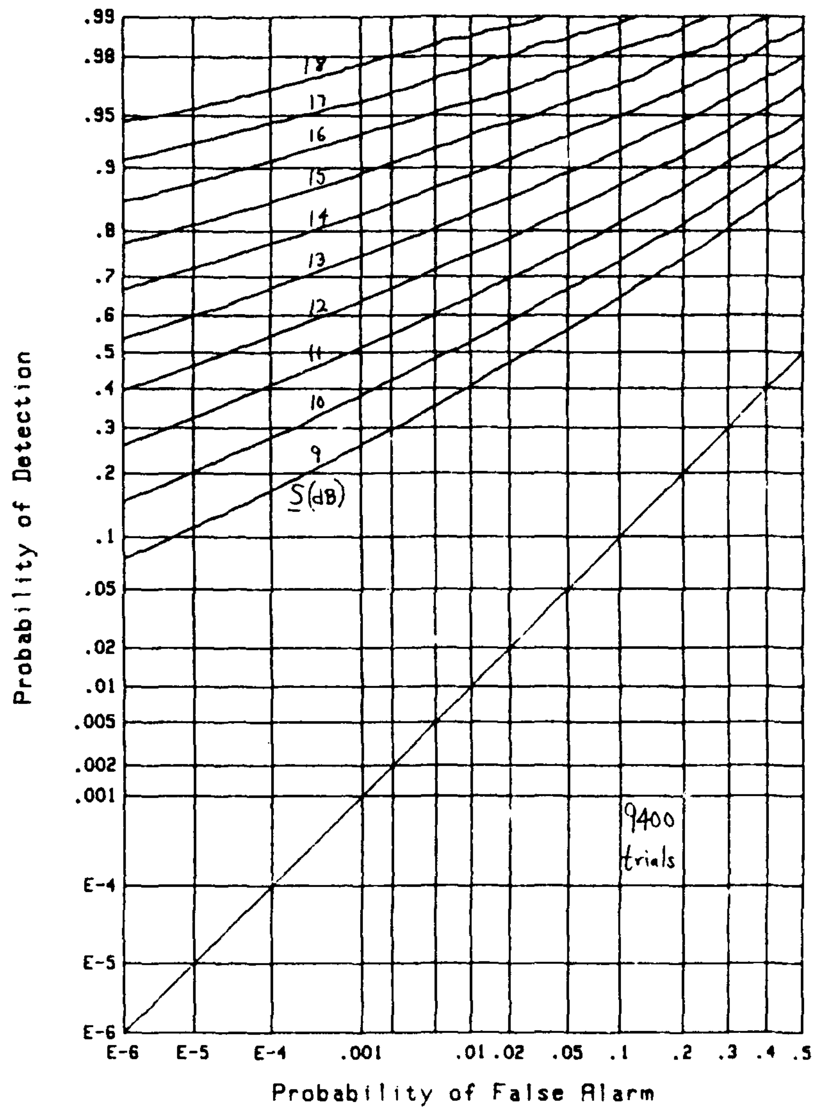

Figure E-26. ROC for SOML, M=3, M=32

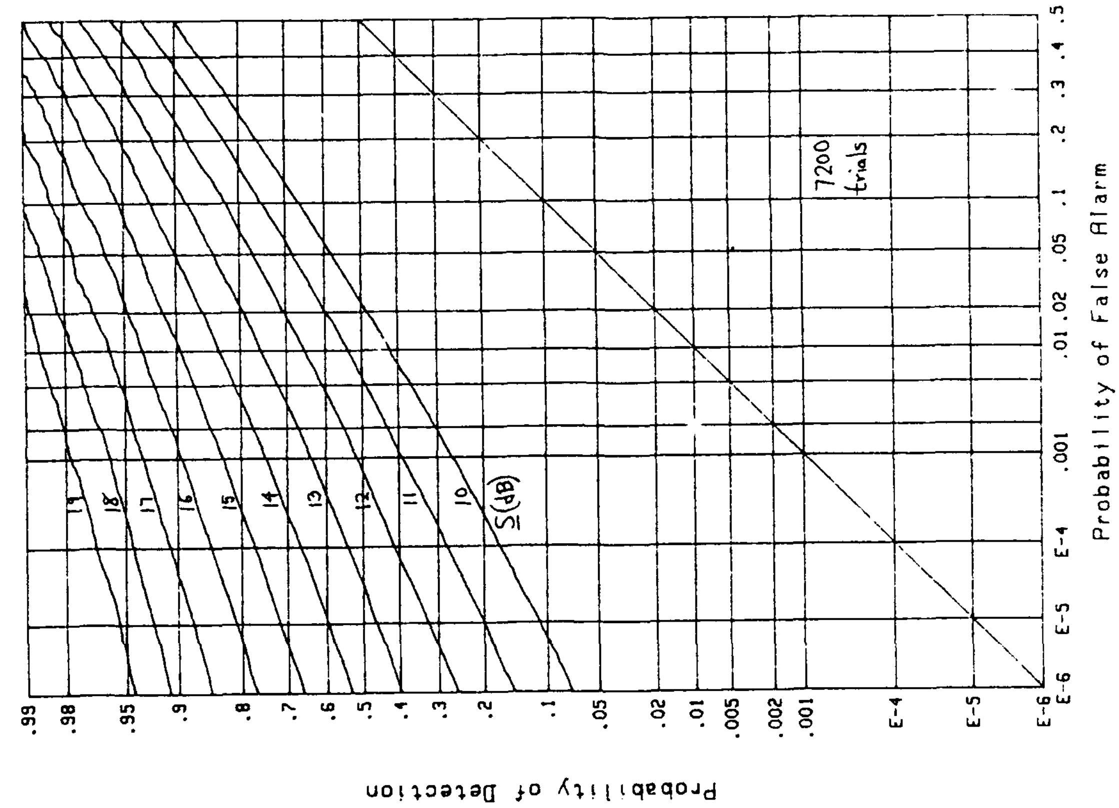

Figure E-27. ROC for SOML, M=3, M=64

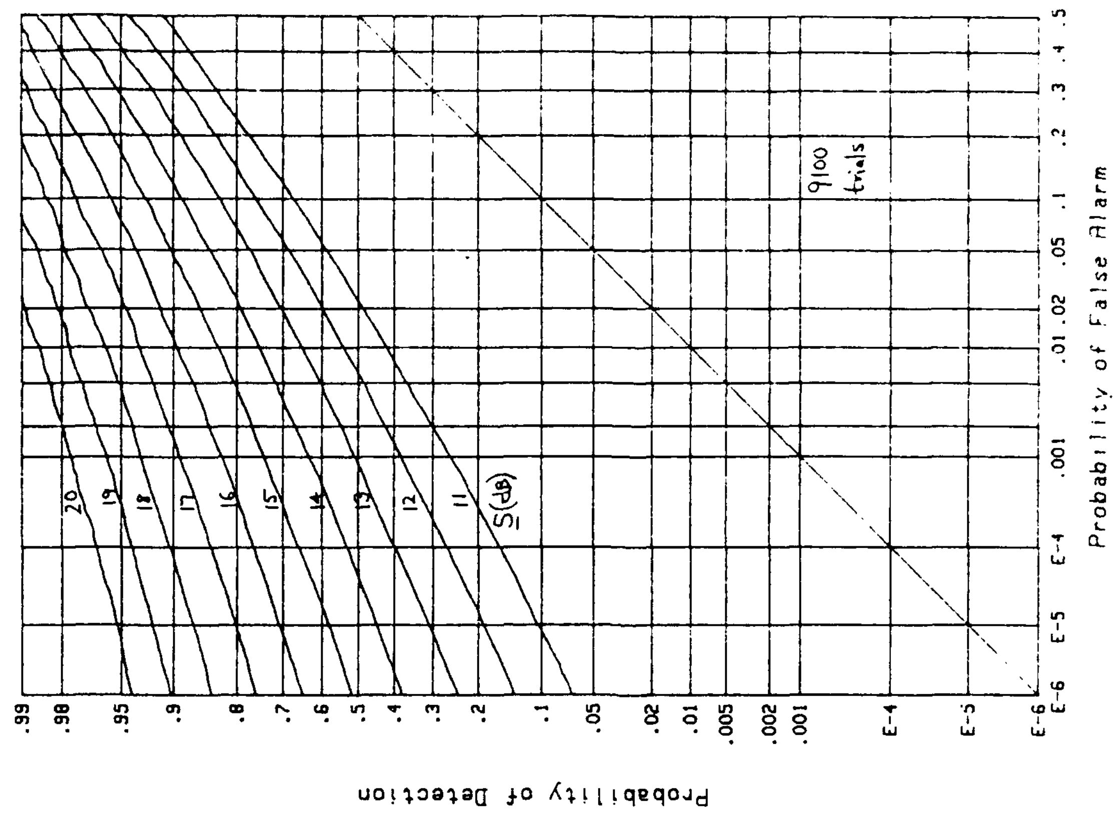

Figure E-28. ROC for SOML, M=3, M=128

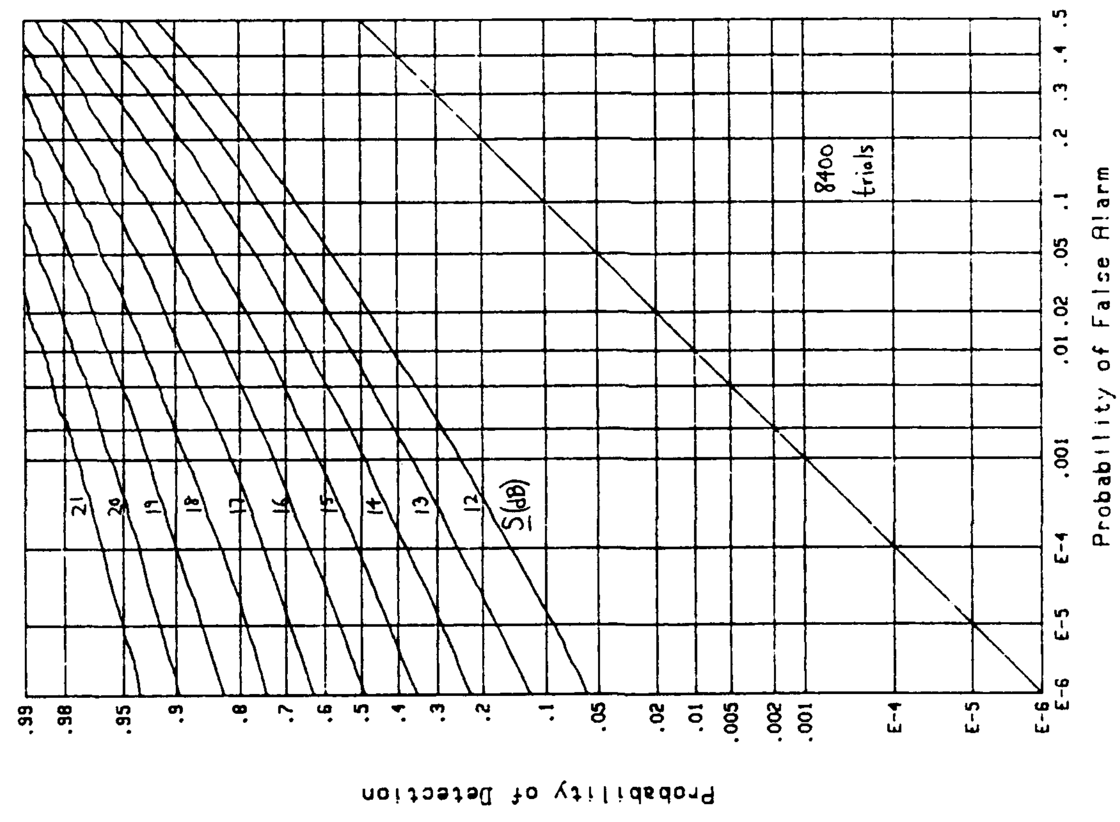

Figure E-29. ROC for SOML, M=3, M=256

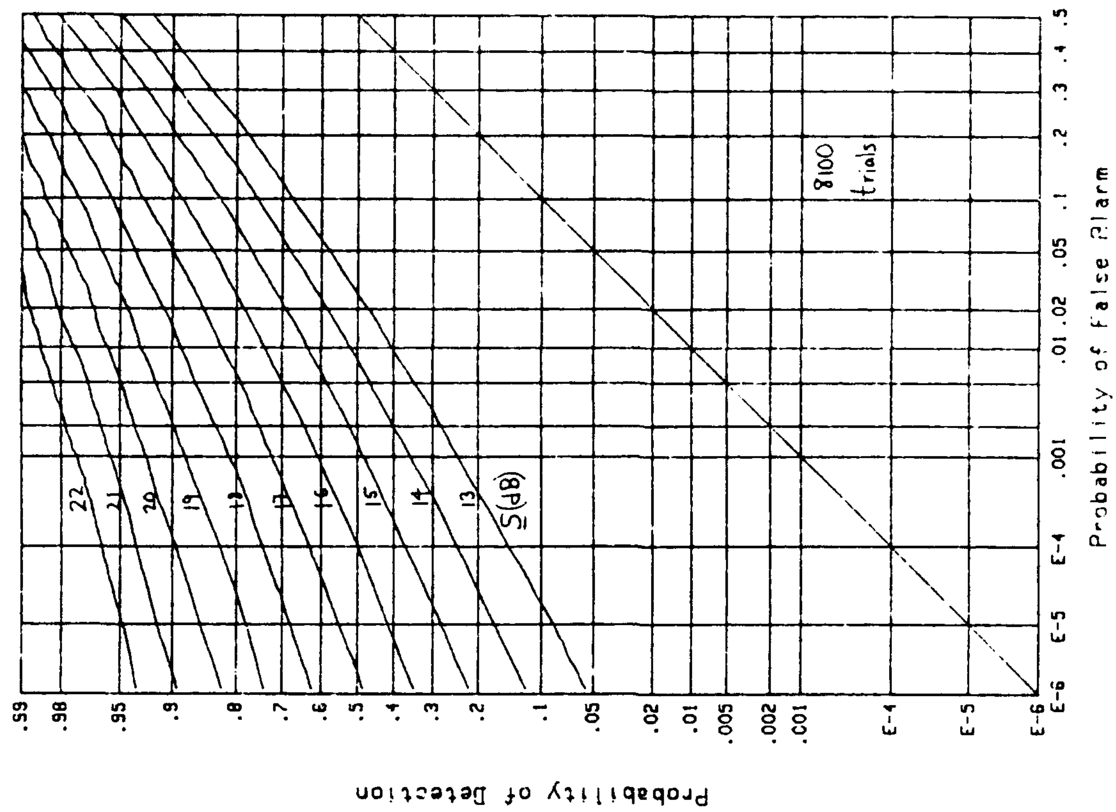

Figure E-30. ROC for SOML, M=3, M=512

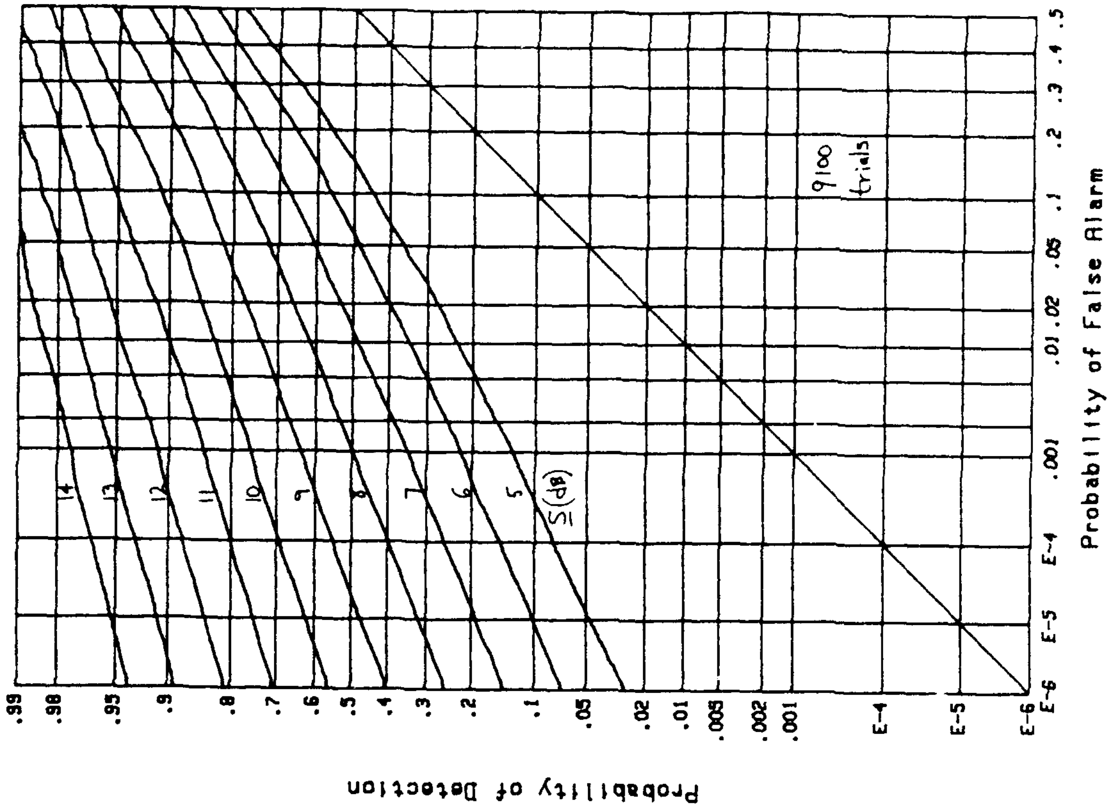

Figure E-32. ROC for SOML, M=4, M=3

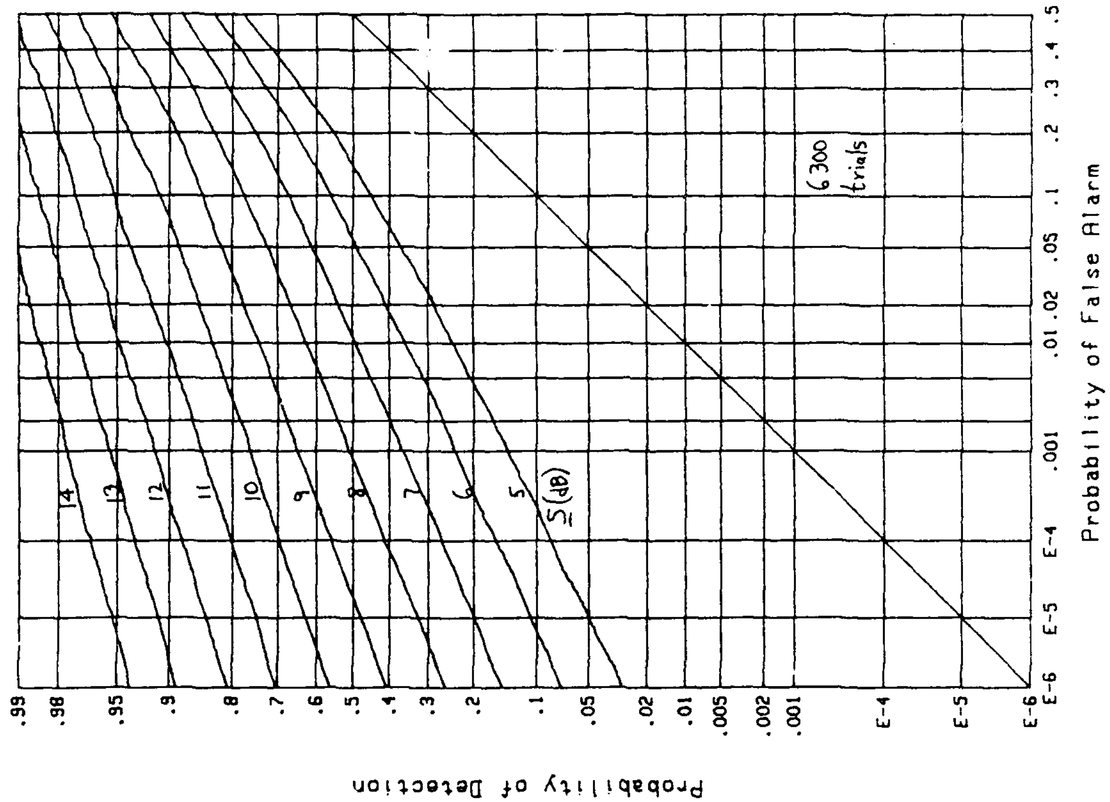

Figure E-31. ROC for SOML, M=4, M=2

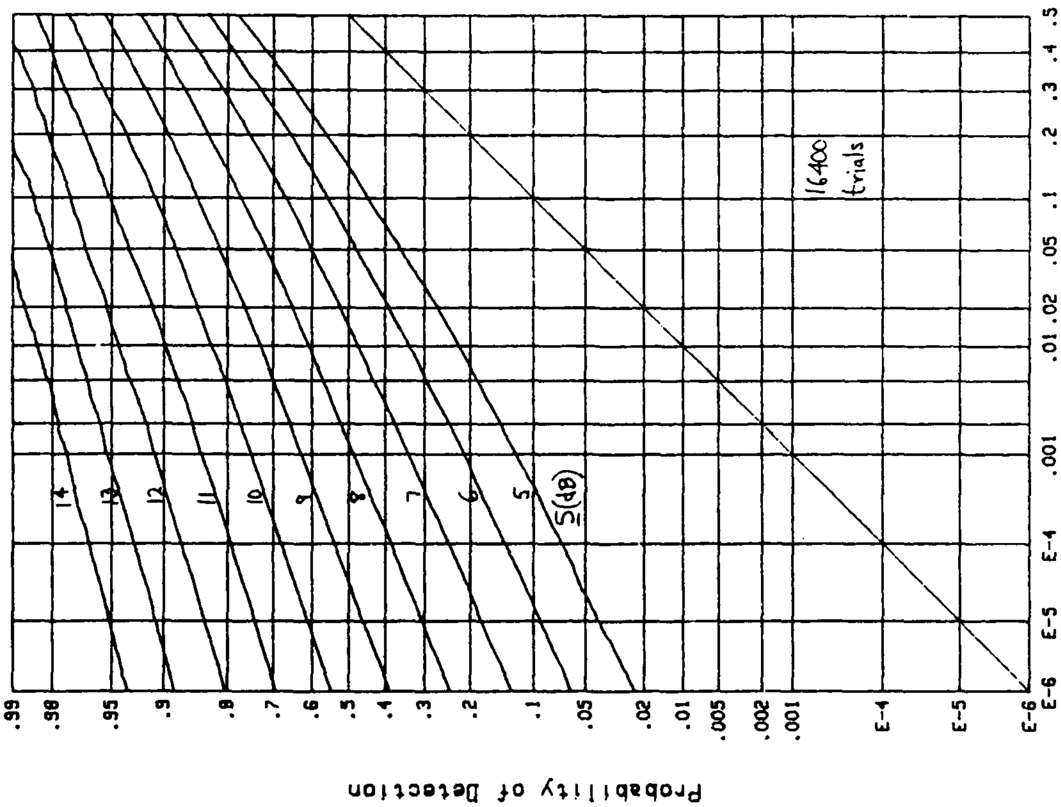

Figure E-33. ROC for SOML, M=4, M=4

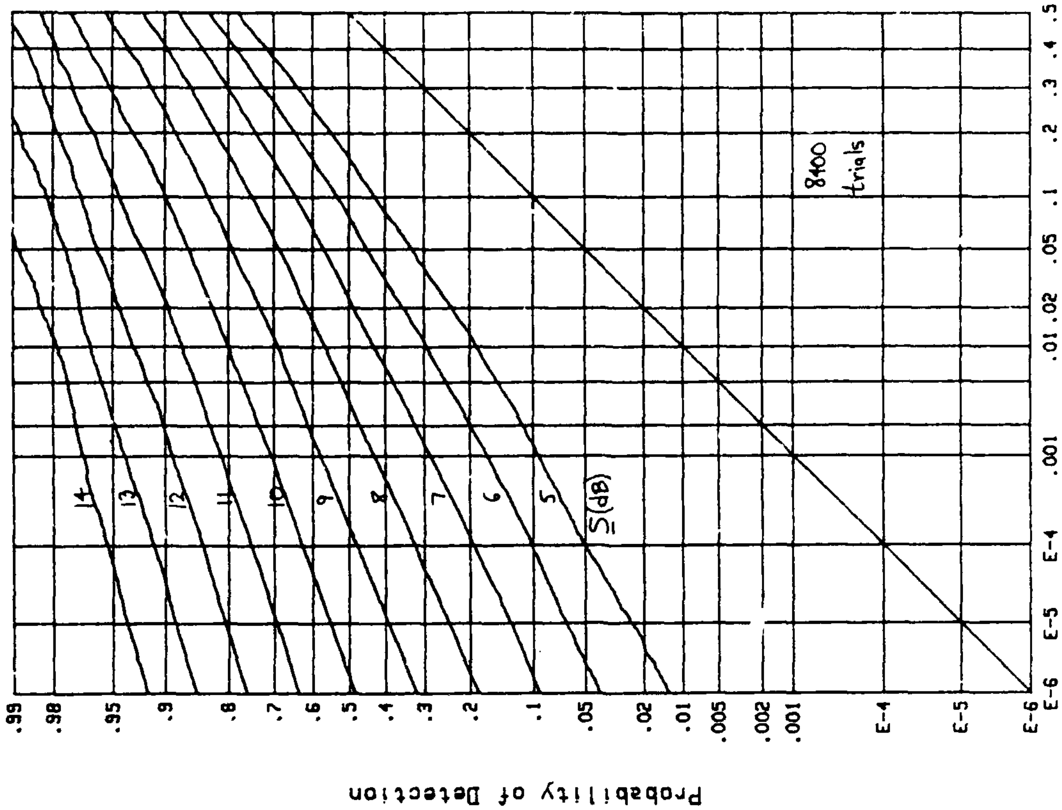

Figure E-34. ROC for SOML, M=4, M=8

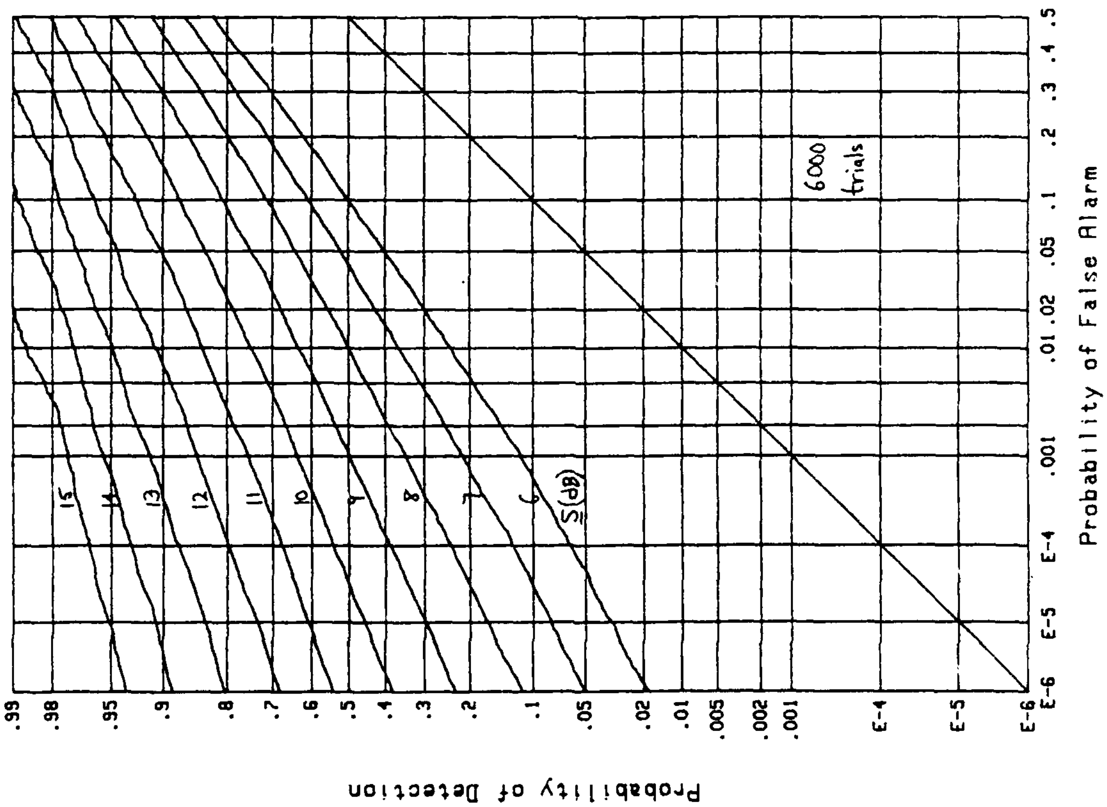

Figure E-35. ROC for SOML, M=4, M=16

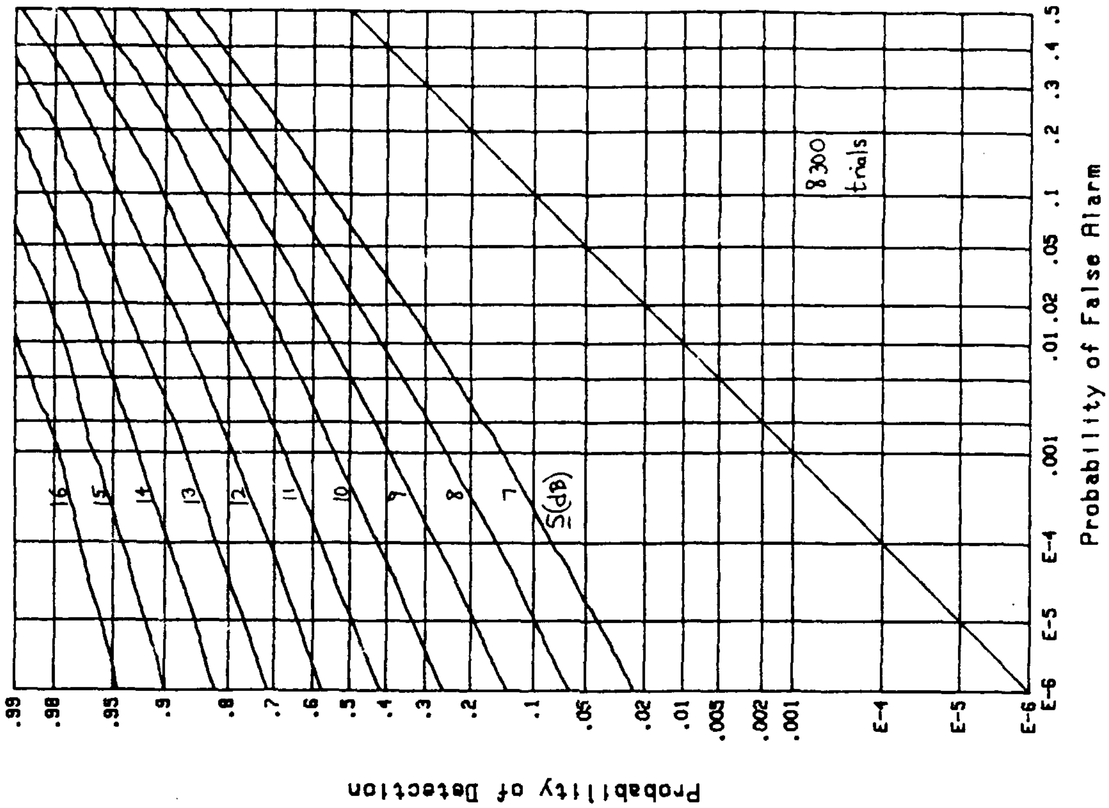

Figure E-36. ROC for SOML, M=4, M=32

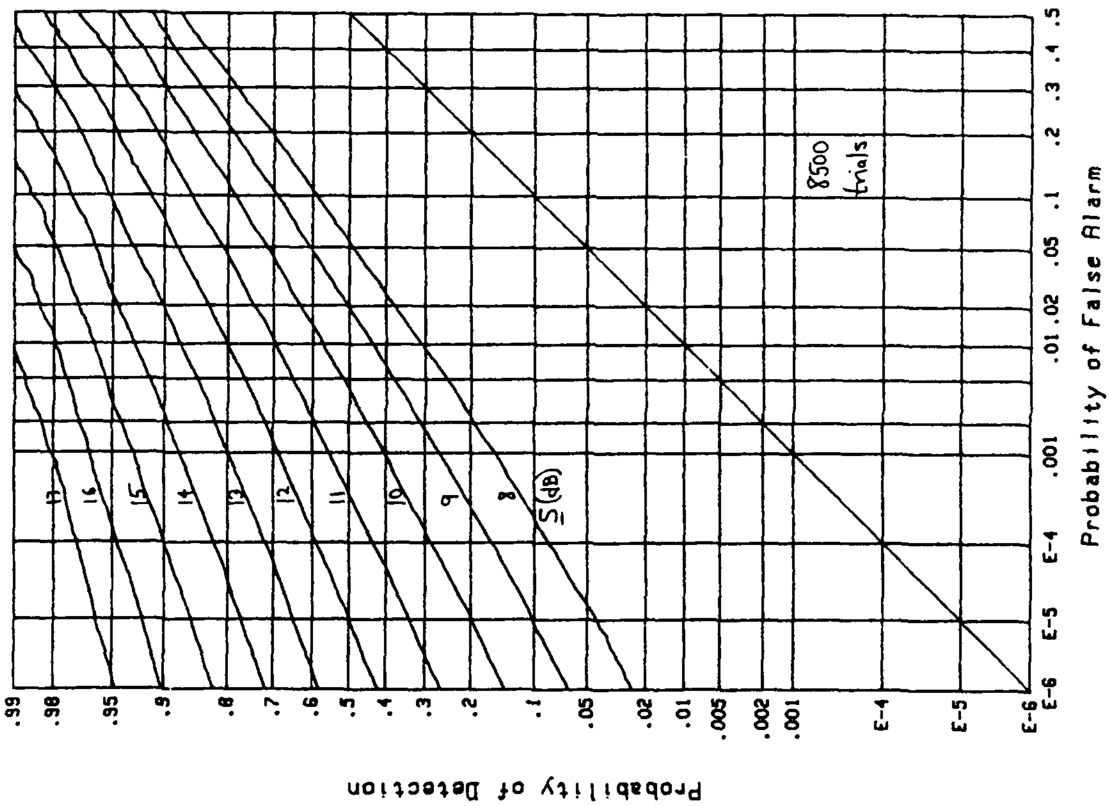

Figure E-37. ROC for SOML, M=4, M=64

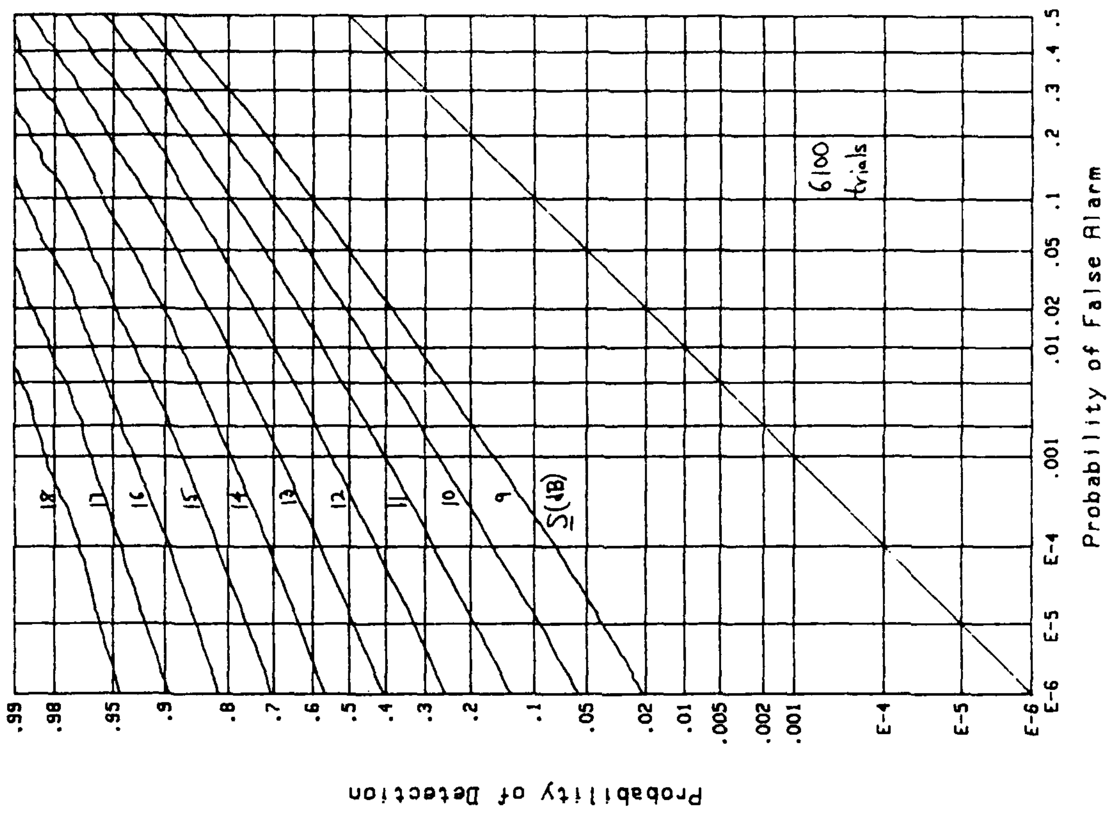

Figure E-38. ROC for SOML, M=4, M=128

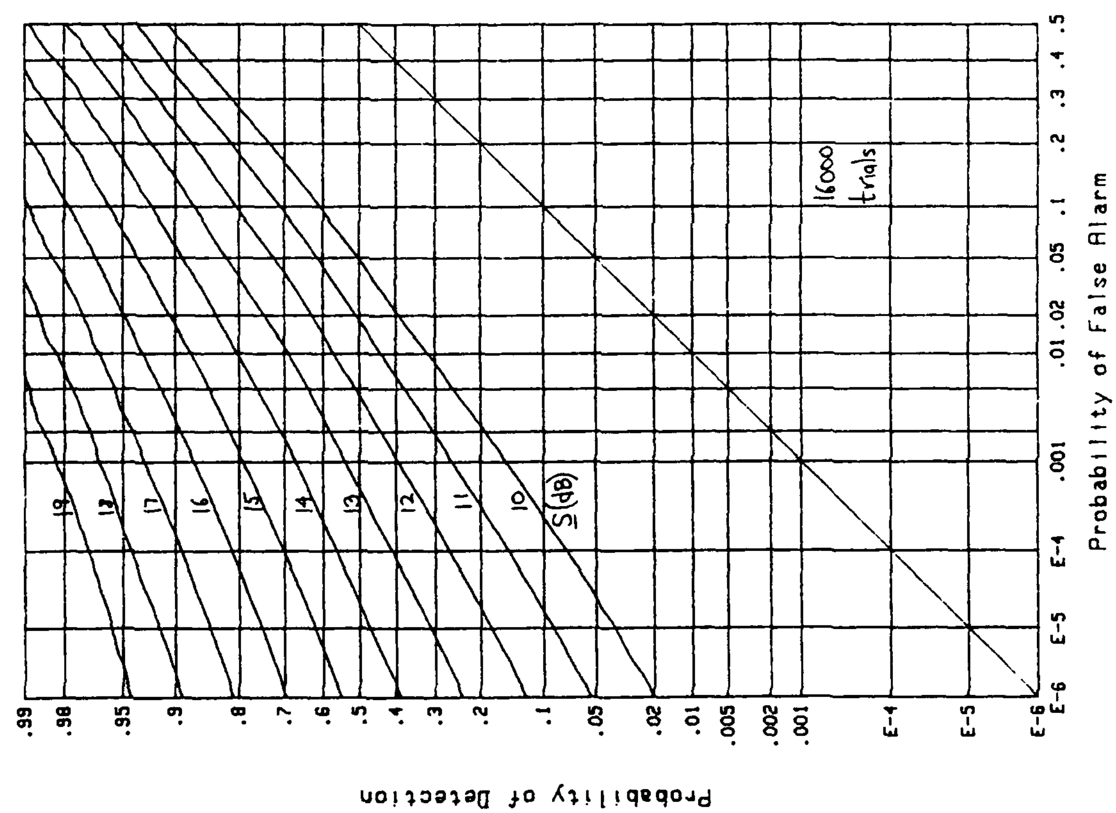

Figure E-39. ROC for SOML, $\underline{M}=4$, M=256

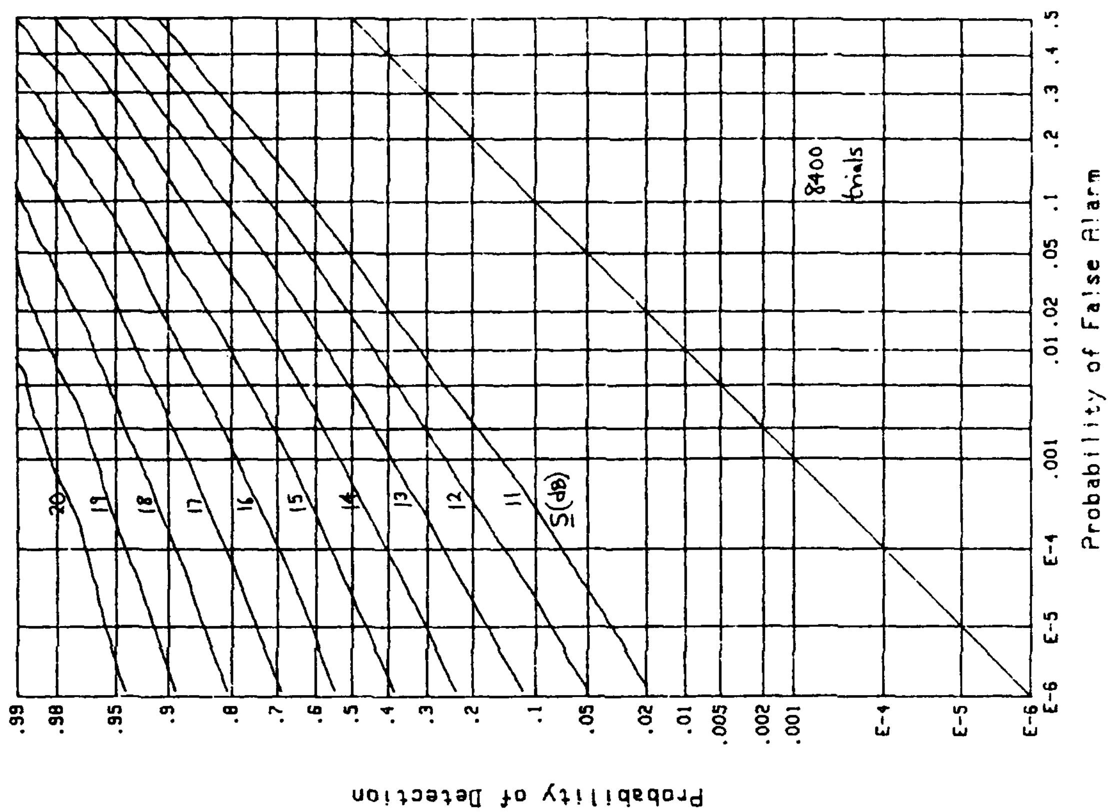

Figure E-40. ROC for SOML, $\underline{M}=4$, M=512

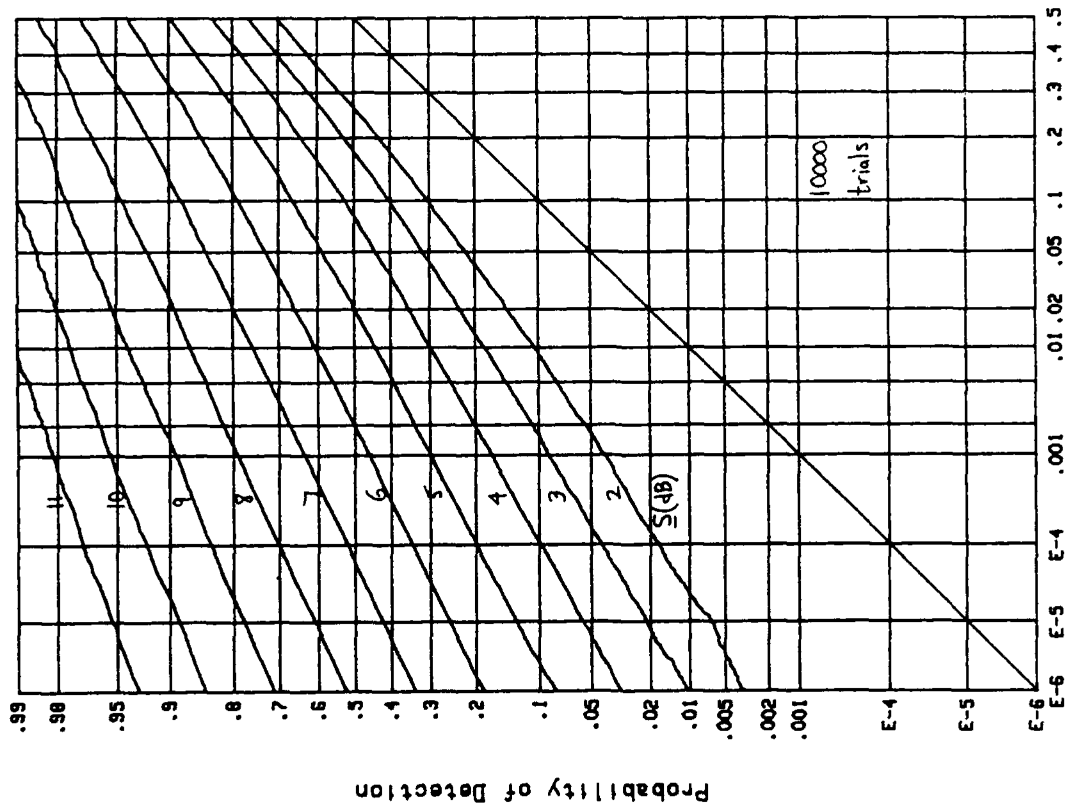

Figure E-41. ROC for SOML, M=8, M=2

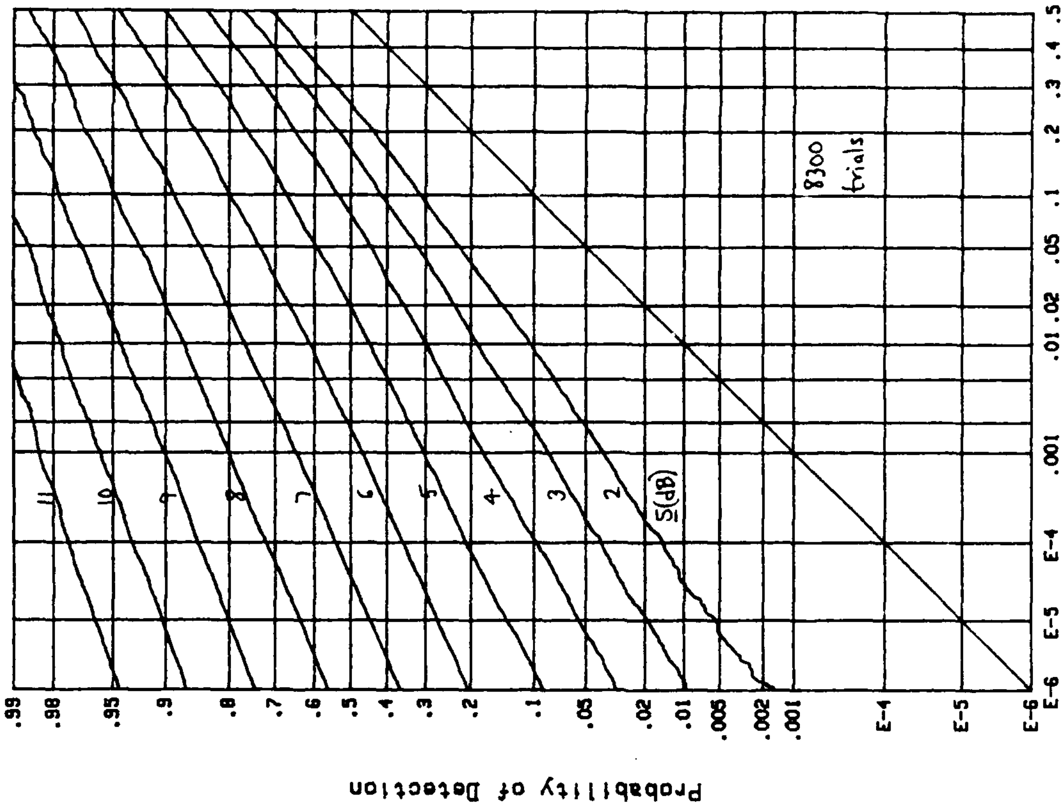

Figure E-42. ROC for SOML, M=8, M=3

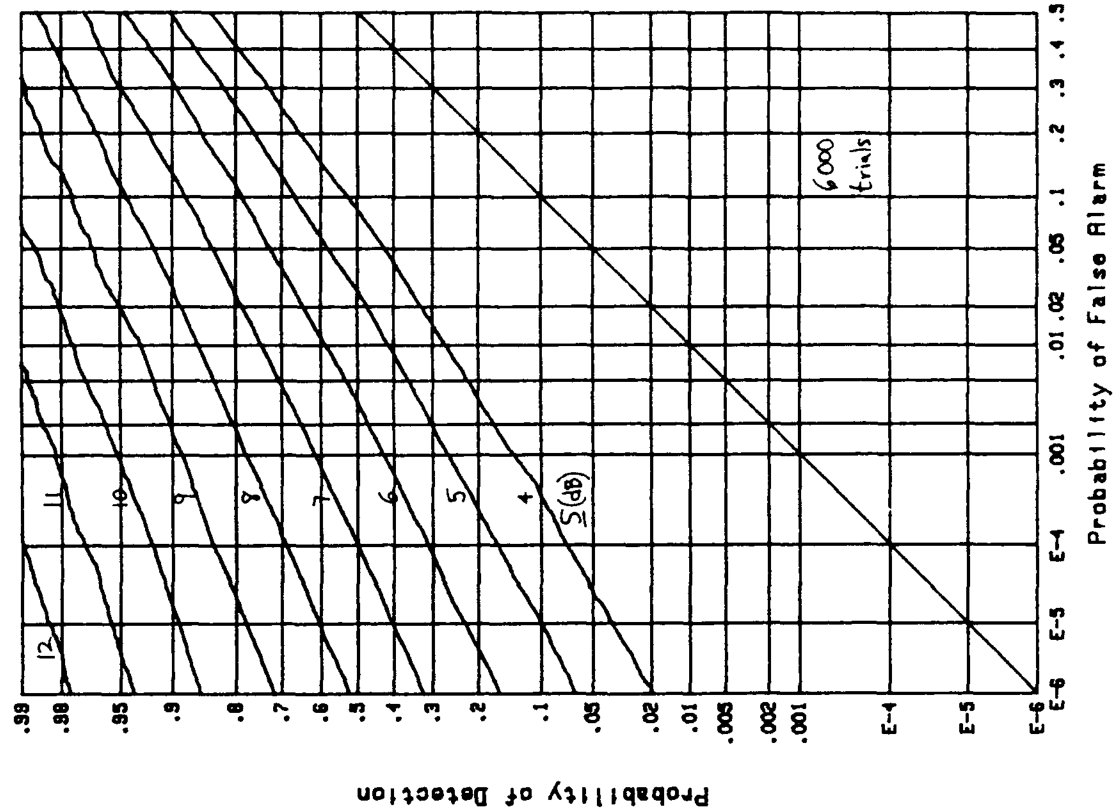

Figure E-44. ROC for SOML, M=8, M=8

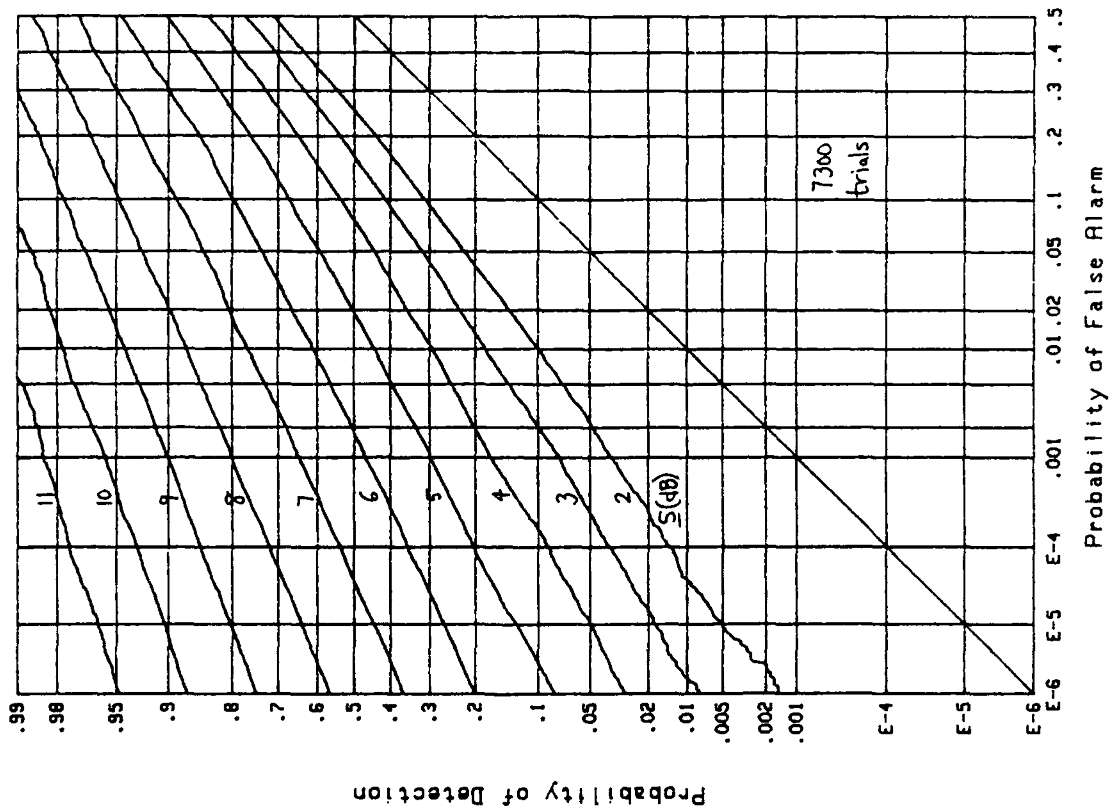

Figure E-43. ROC for SOML, M=8, M=4

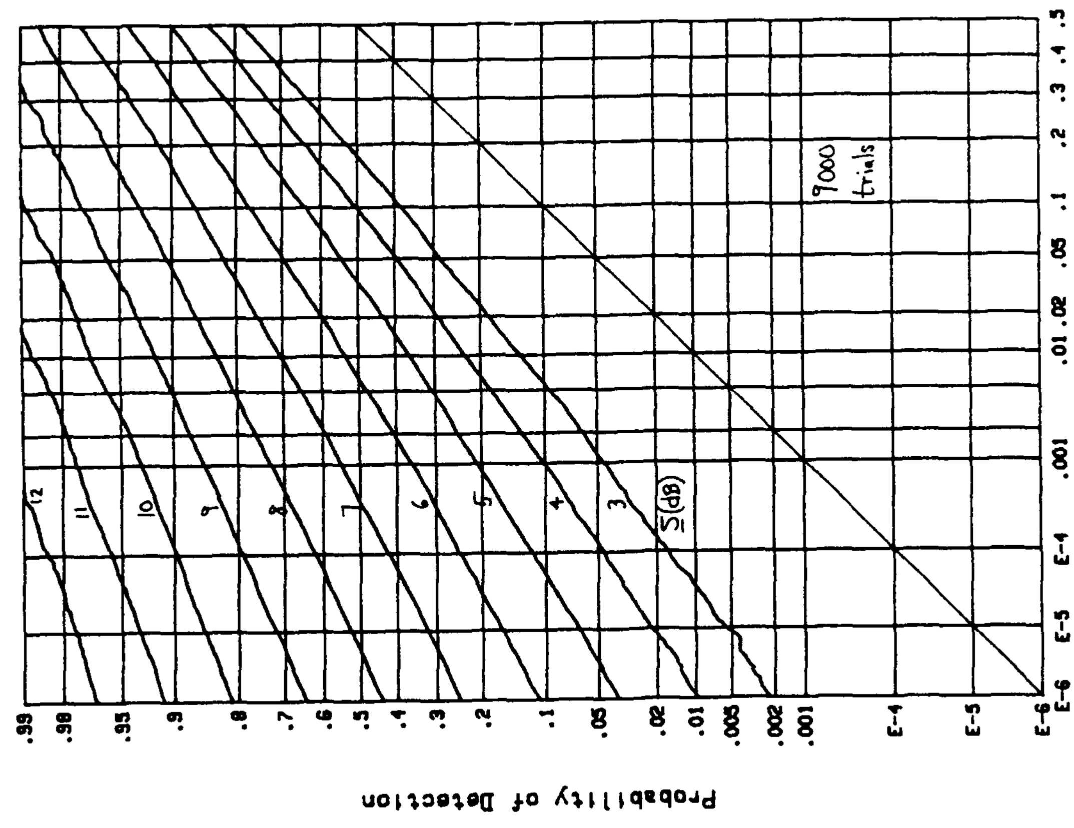

Figure E-45. ROC for SOML, M=8, M=16

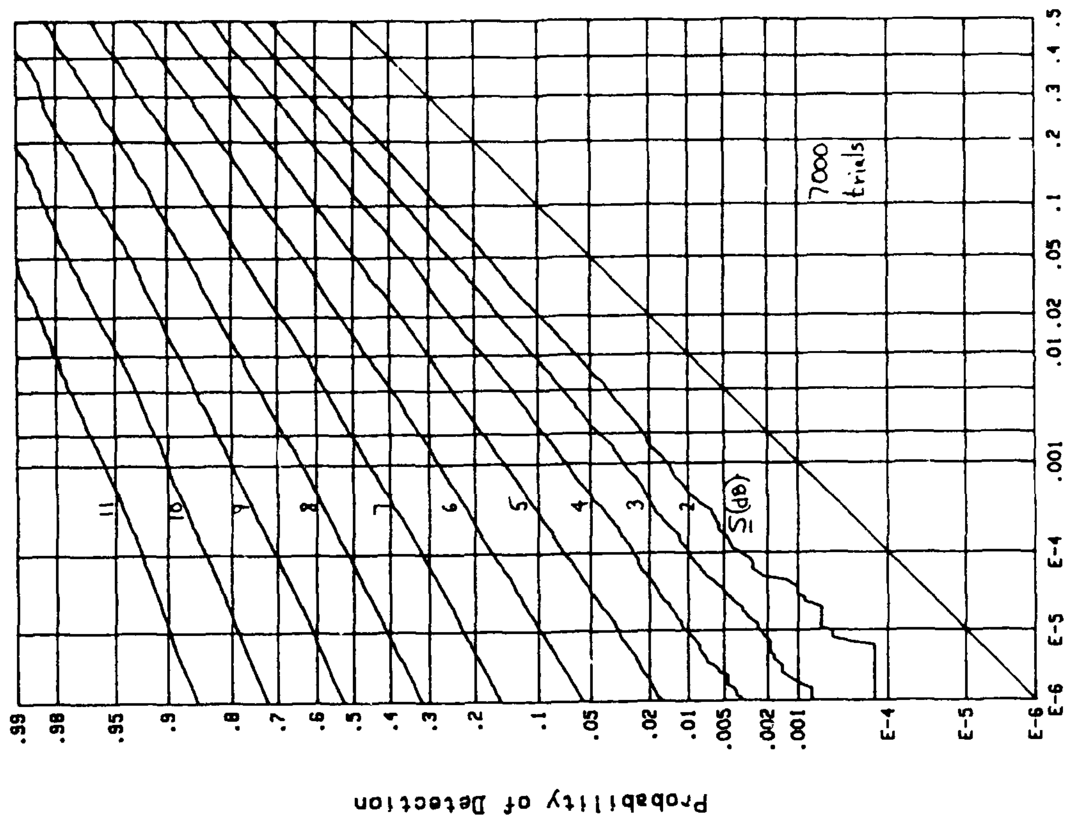

Figure E-46. ROC for SOML, M=8, M=32

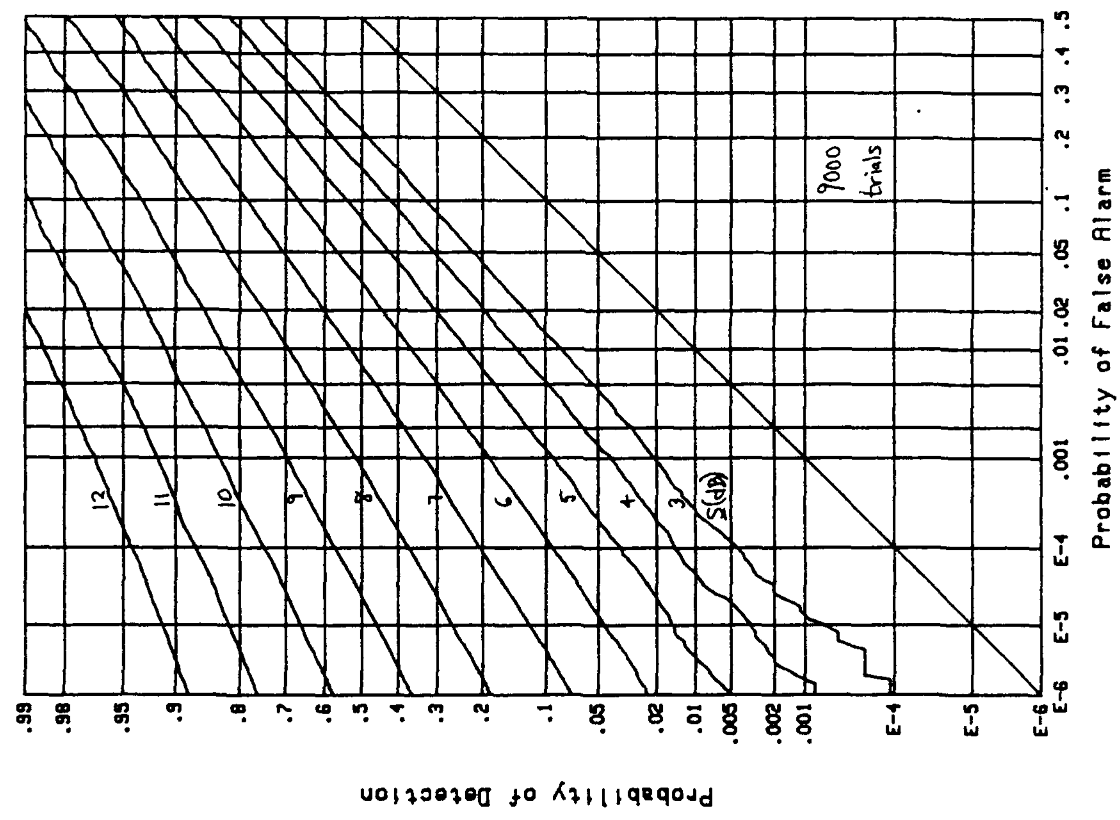

Figure E-47. ROC for SOML, M̲=8, M=64

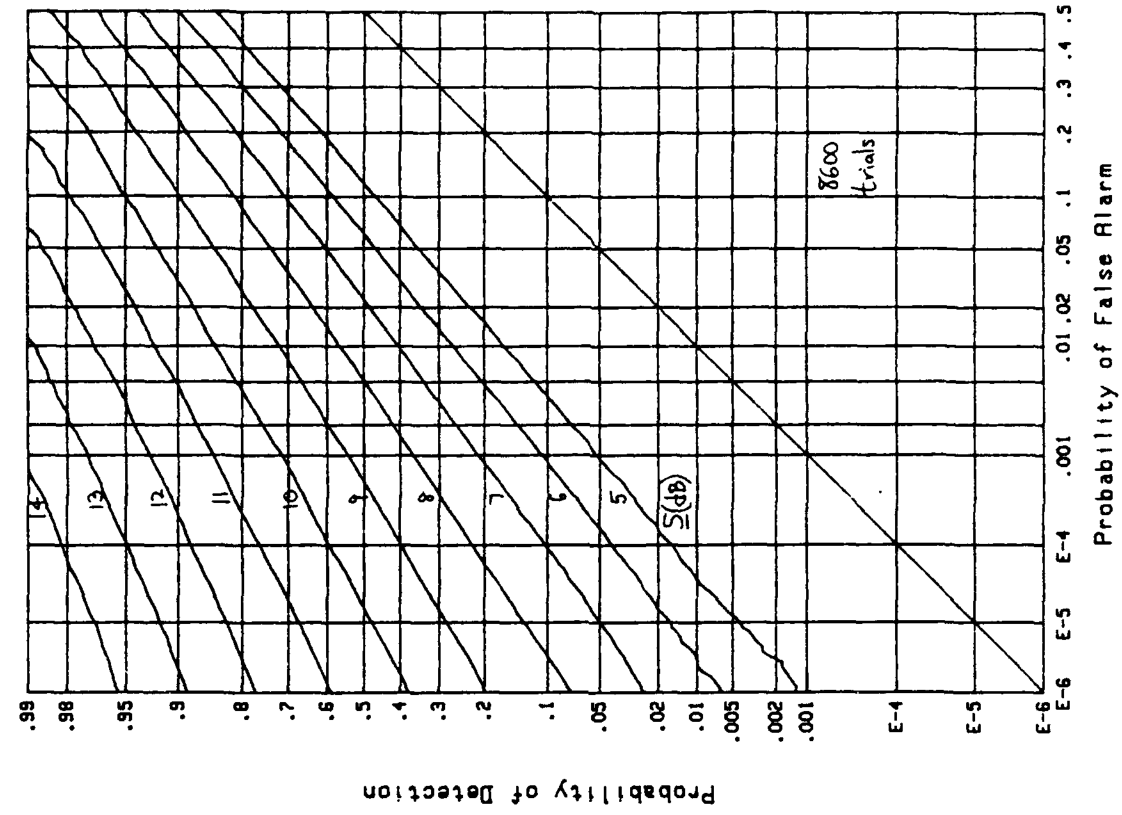

Figure E-48. ROC for SOML, M̲=8, M=128

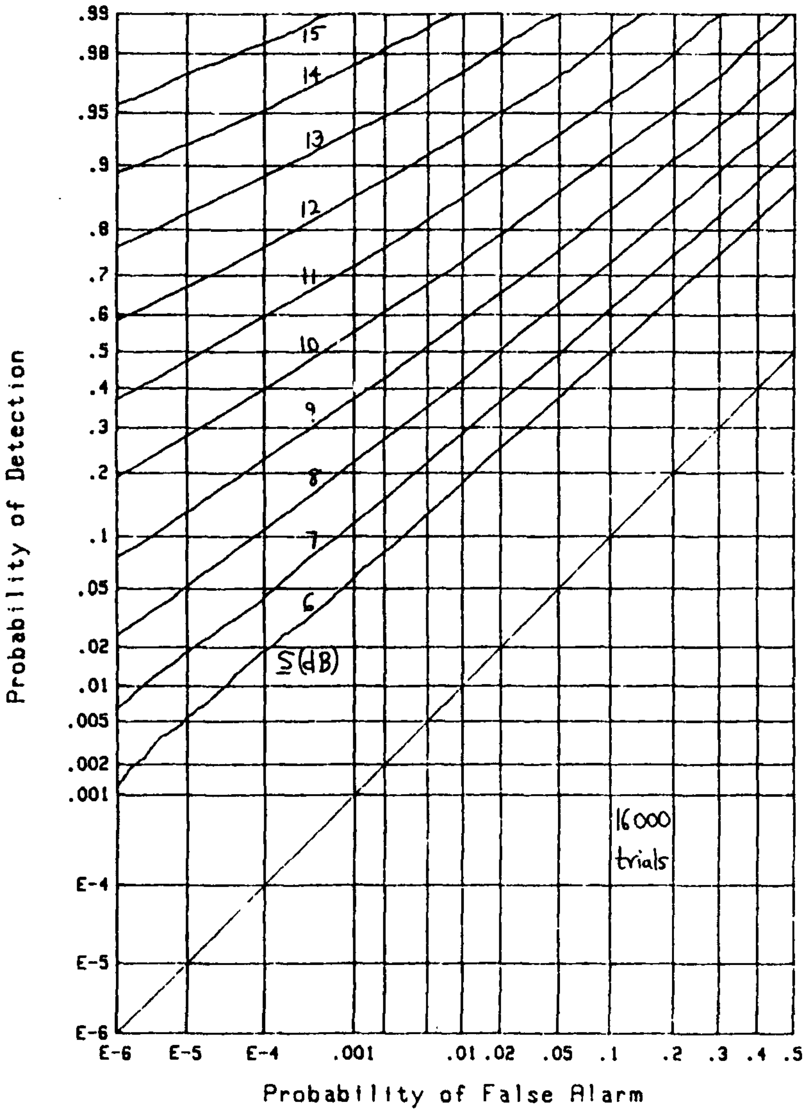

Figure E-49. ROC for SOML, M=8, M=256

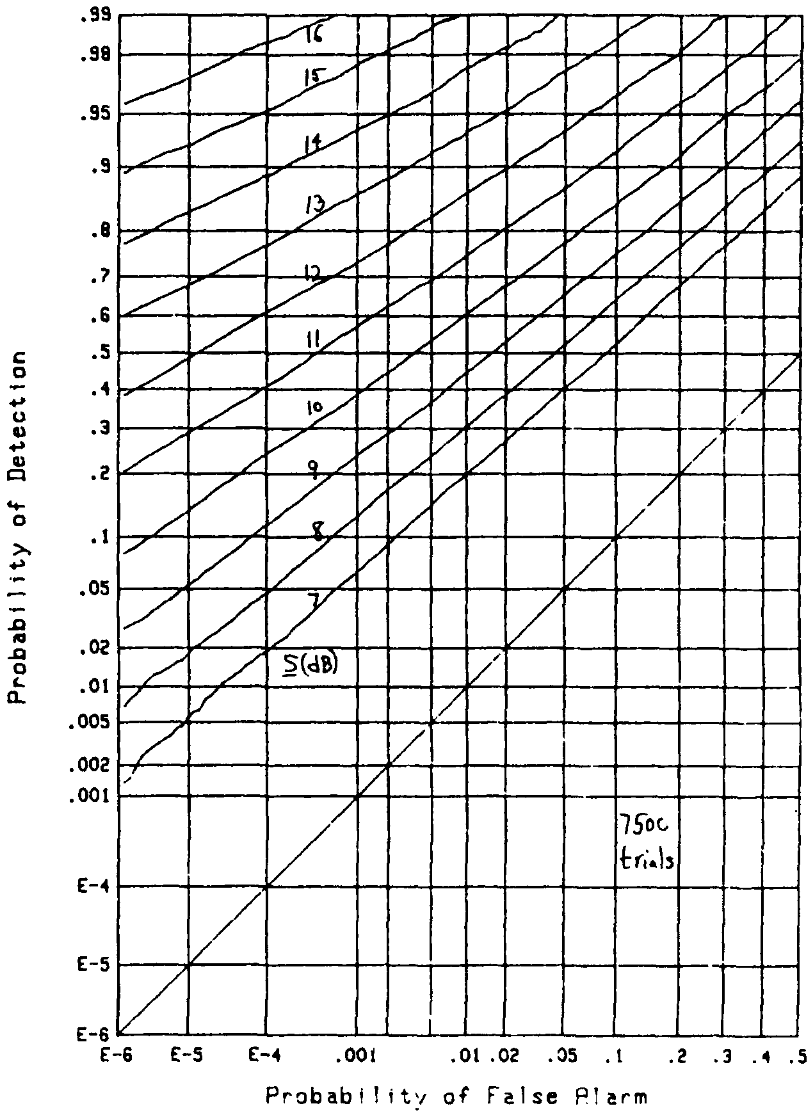

Figure E-50. ROC for SOML, M=8, M=512

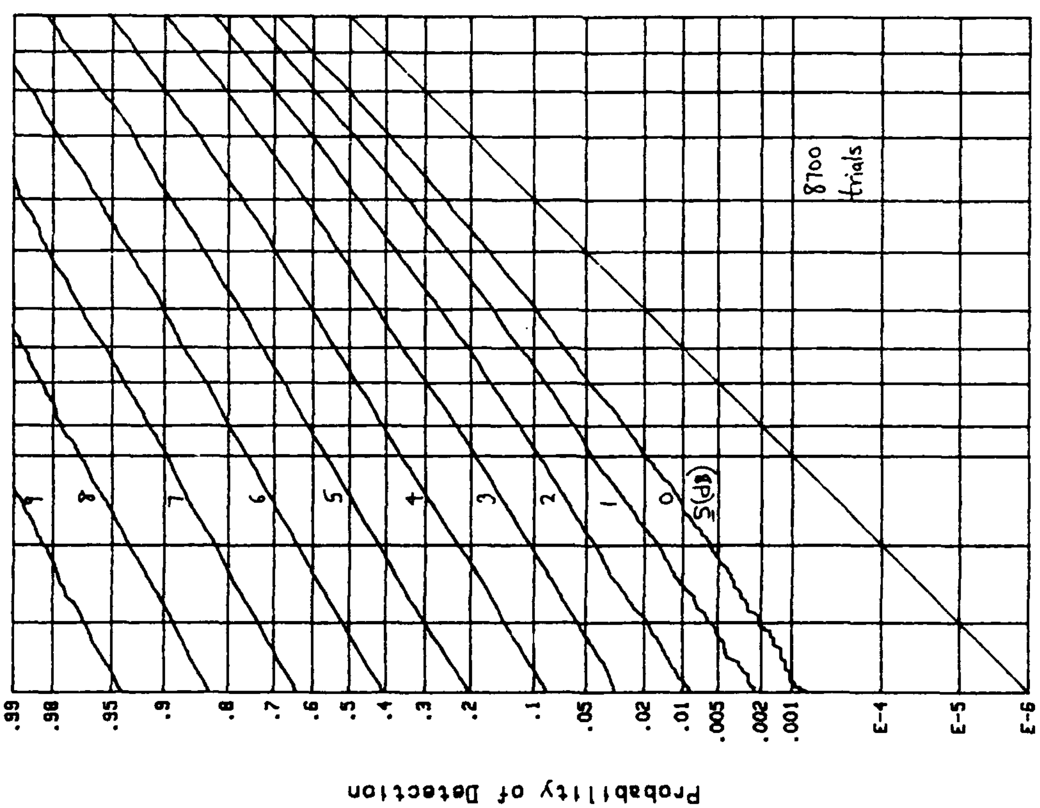

Figure E-51. ROC for SOML, M=16, M=2

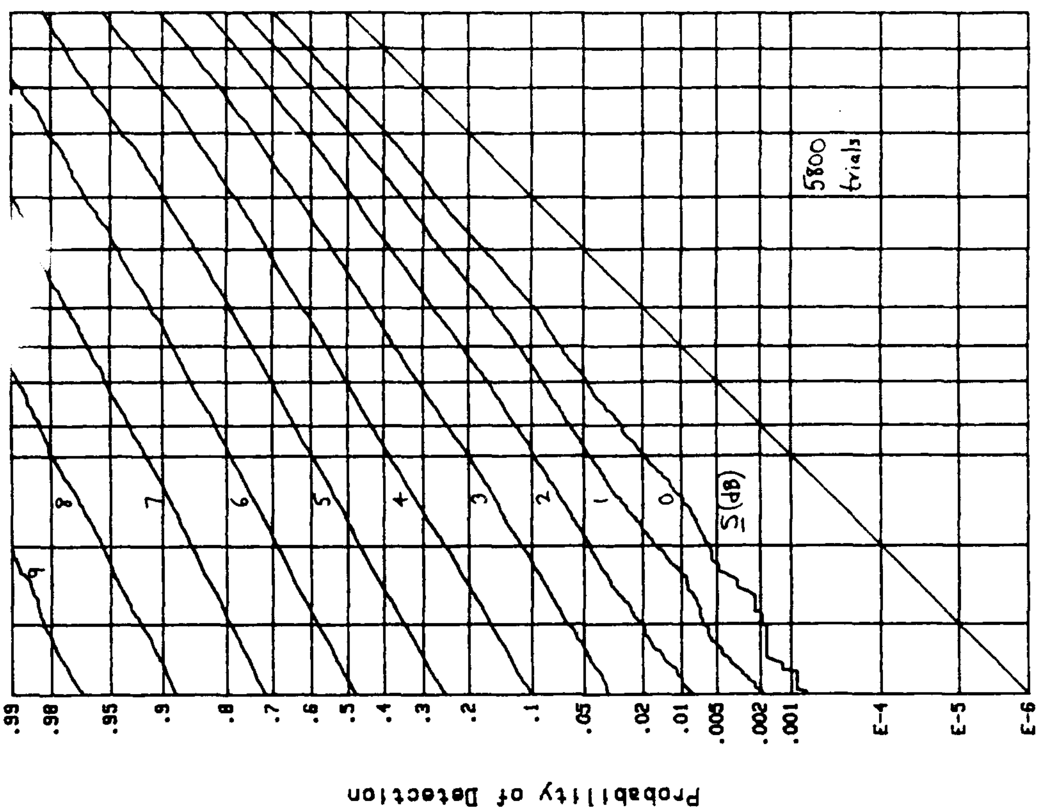

Figure E-52. ROC for SOML, M=16, M=3

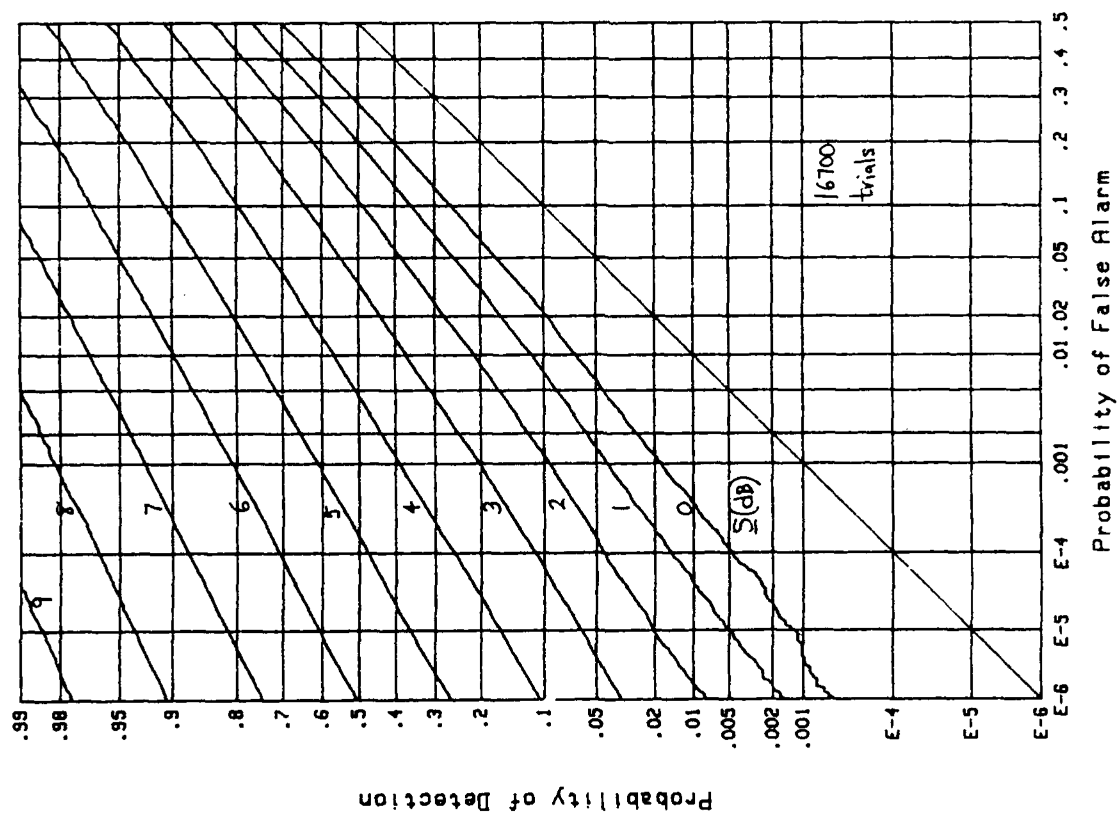

Figure E-53. ROC for SOML, M=16, M=4

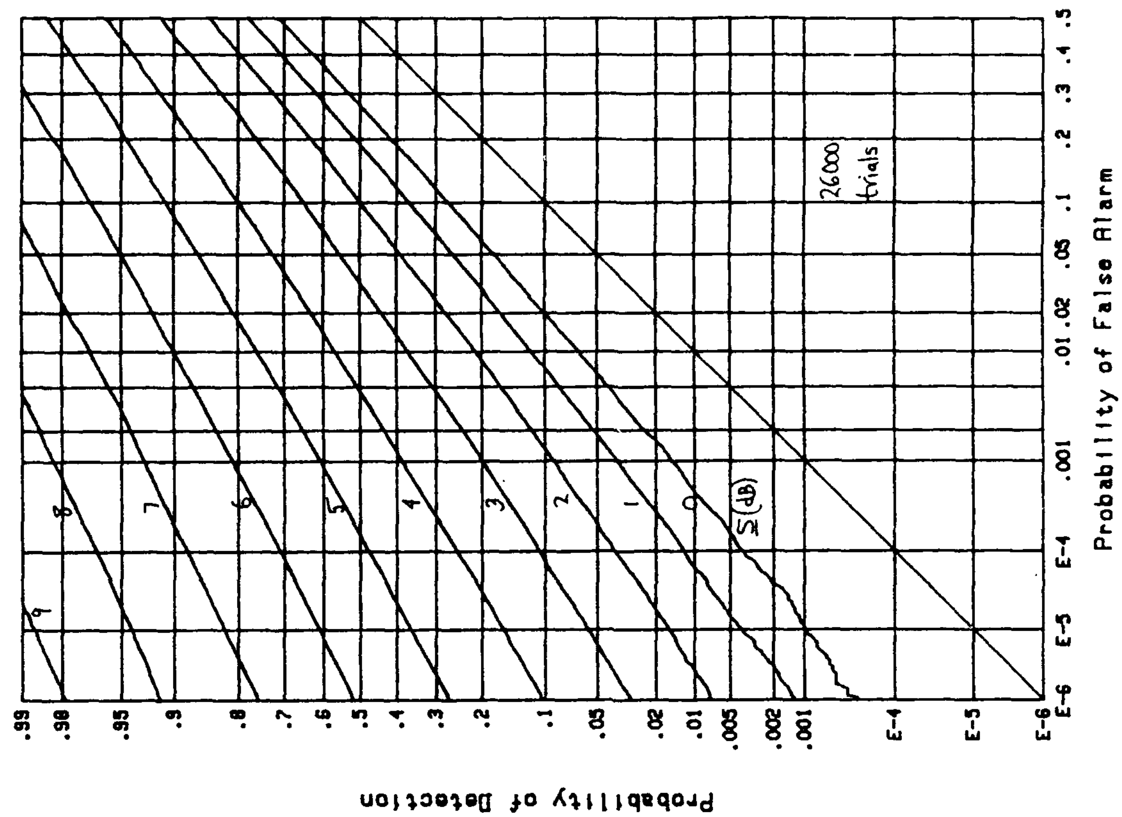

Figure E-54. ROC for SOML, M=16, M=8

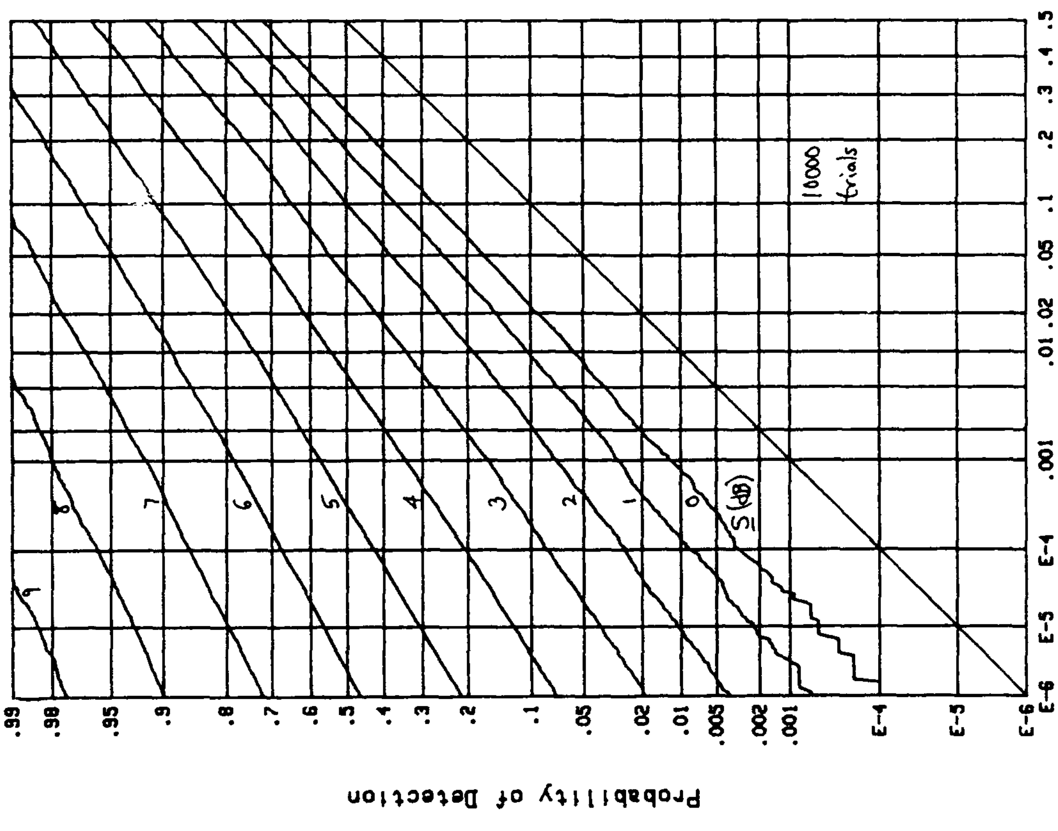

Figure E-55. ROC for SOML, M=16, M=16

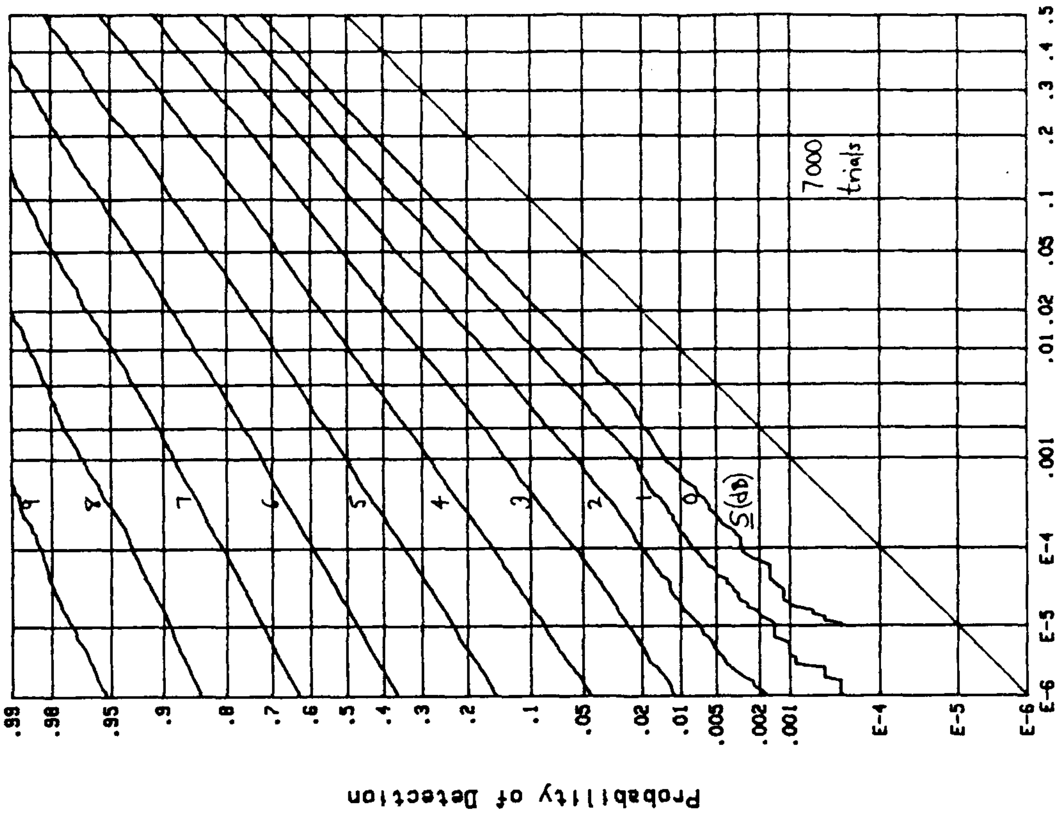

Figure E-56. ROC for SOML, M=16, M=32

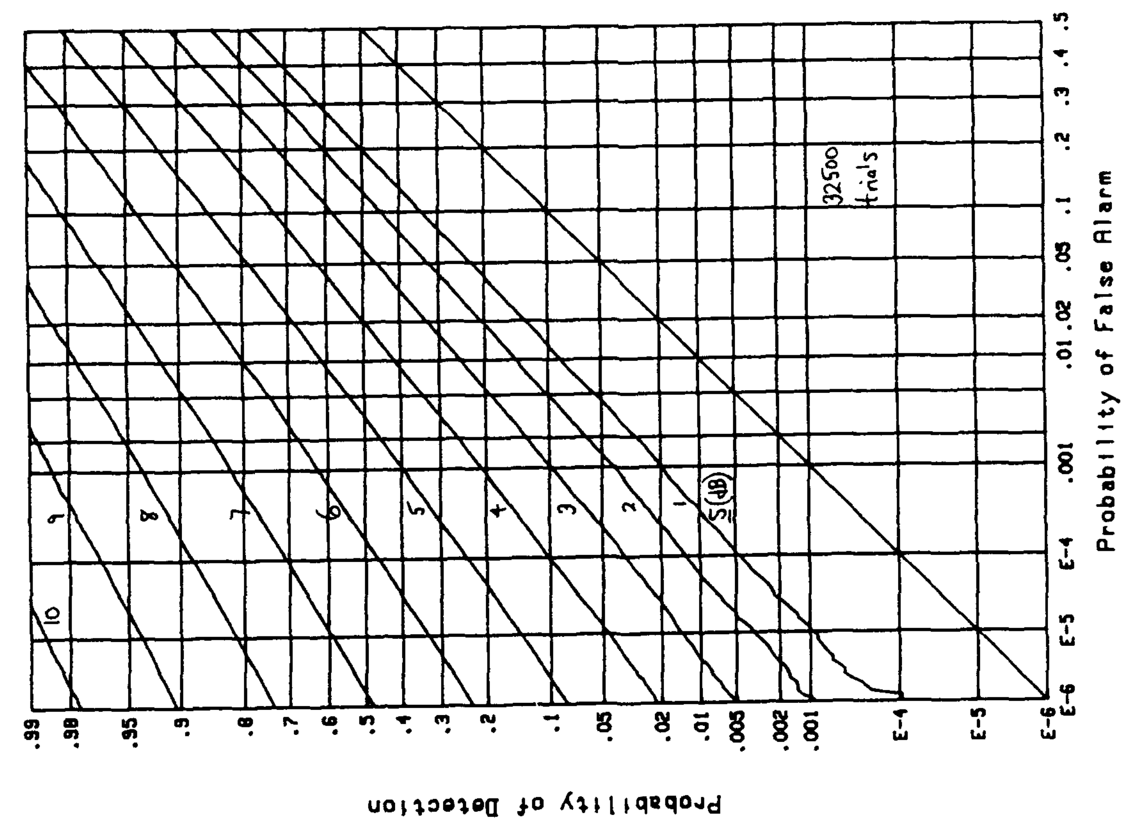

Figure E-57. ROC for SOML, M=16, M=64

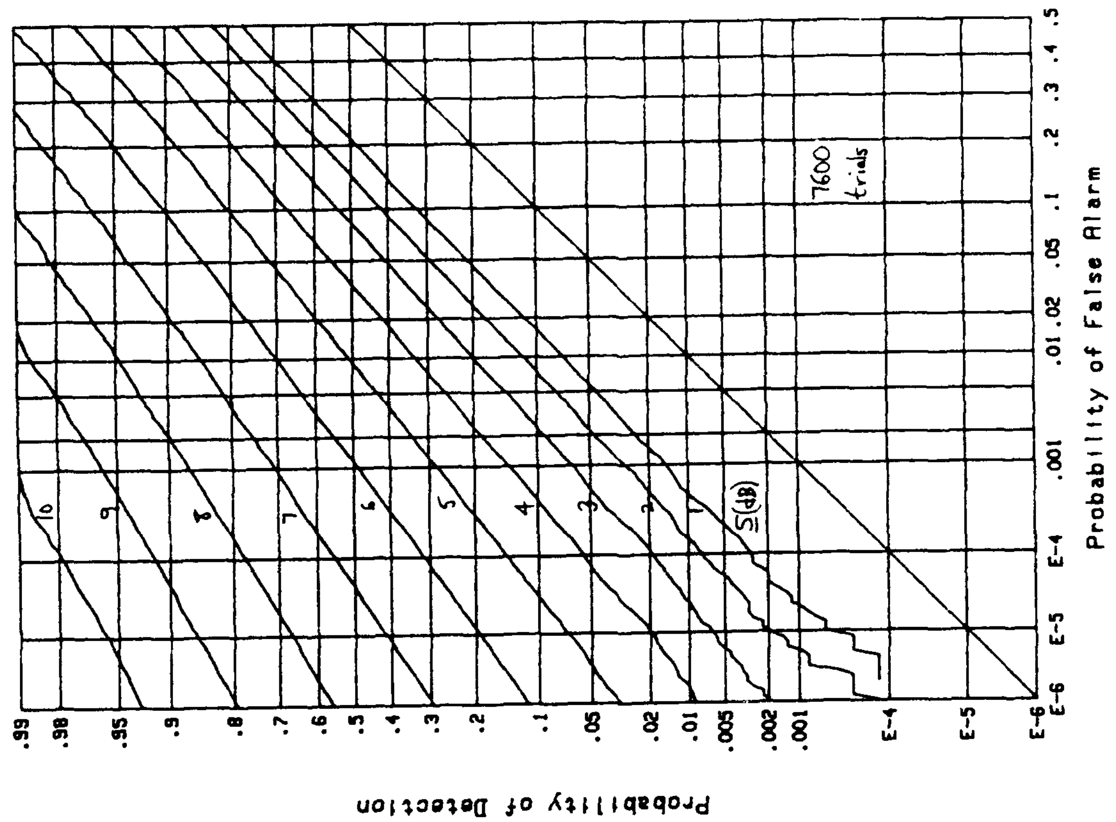

Figure E-58. ROC for SOML, M=16, M=128

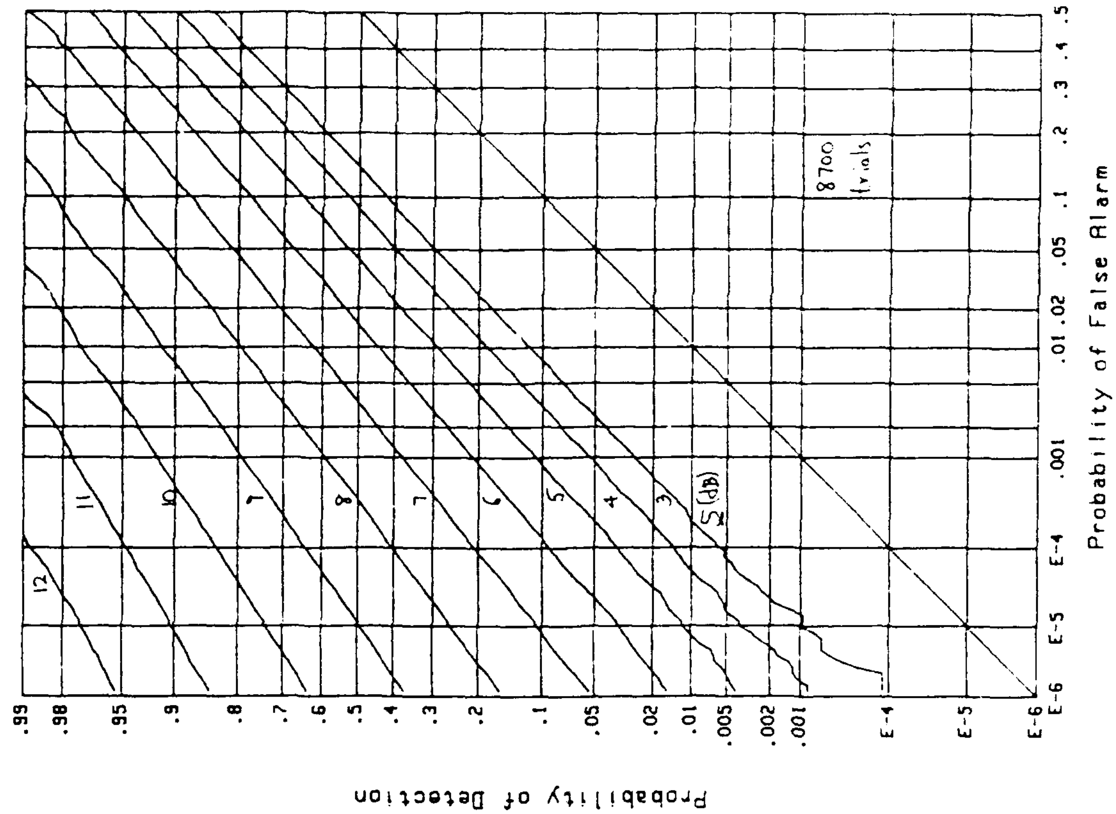

Figure E-60. ROC for SOML, $\underline{M}$=16, M=512

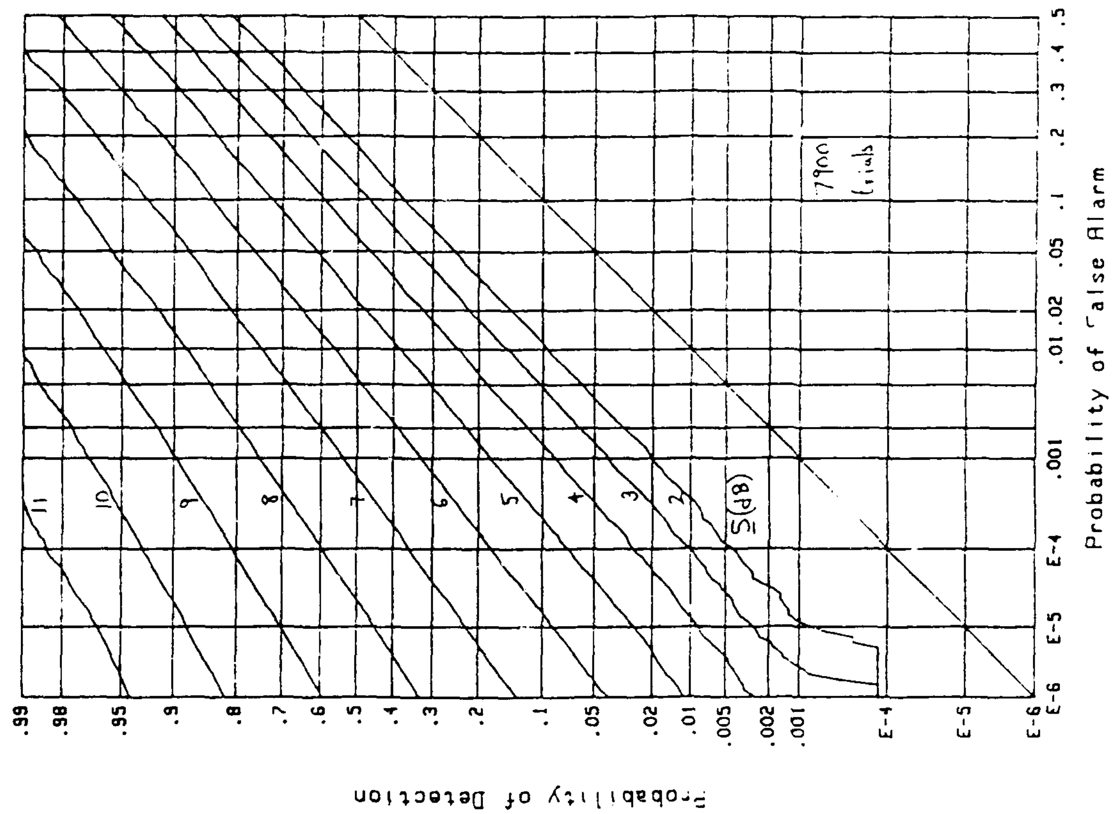

Figure E-59. ROC for SOML, $\underline{M}$=16, M=256

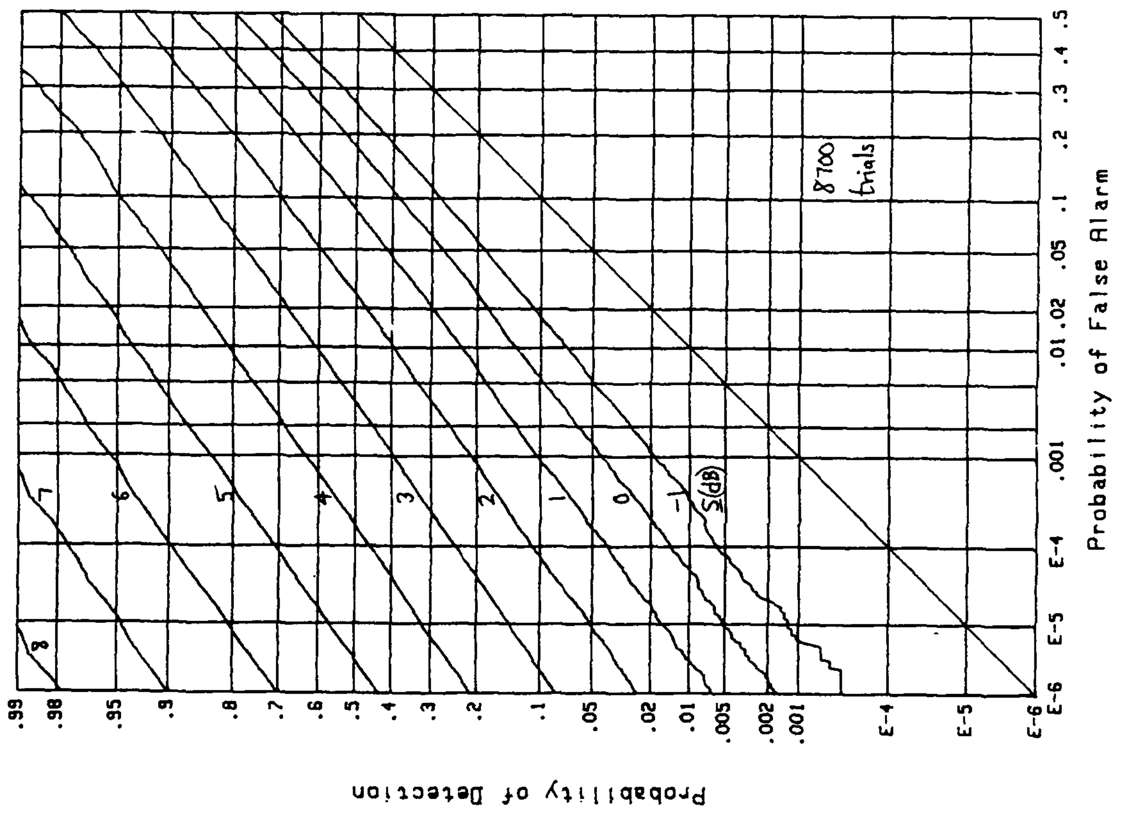

Figure E-61. ROC for SOML, $\underline{M}$=32, M=2

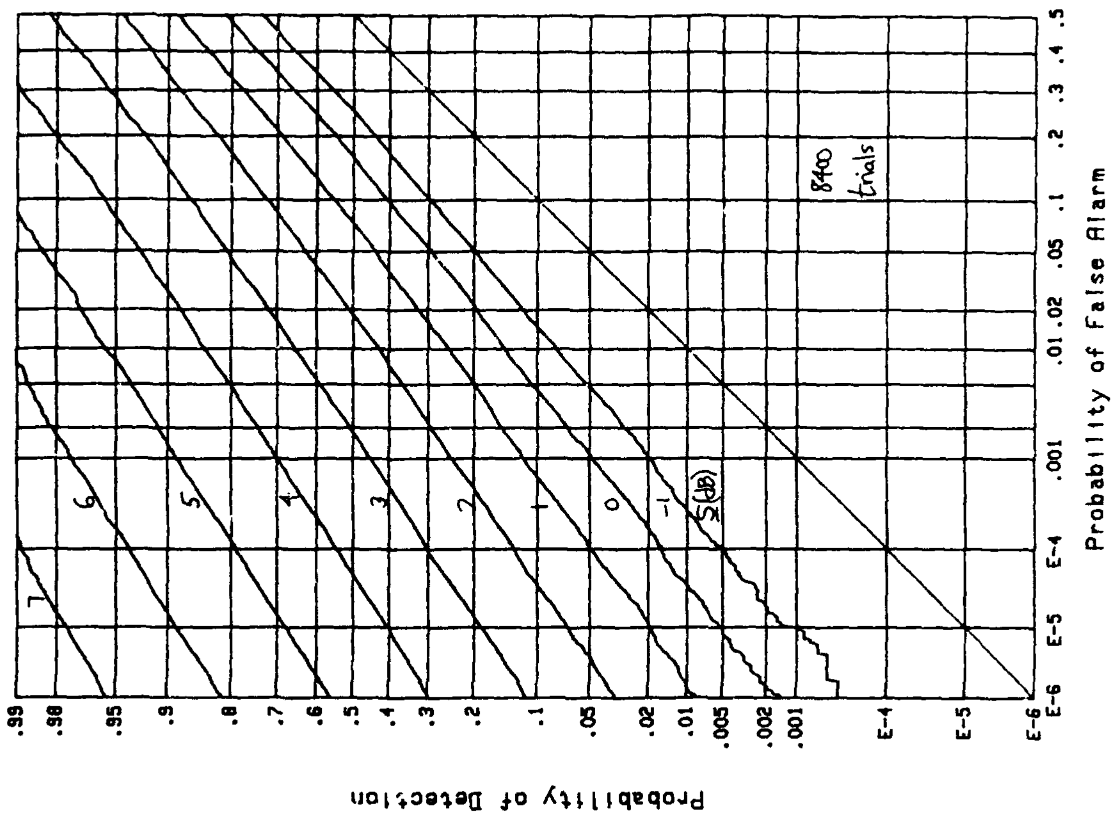

Figure E-62. ROC for SOML, $\underline{M}$=32, M=3

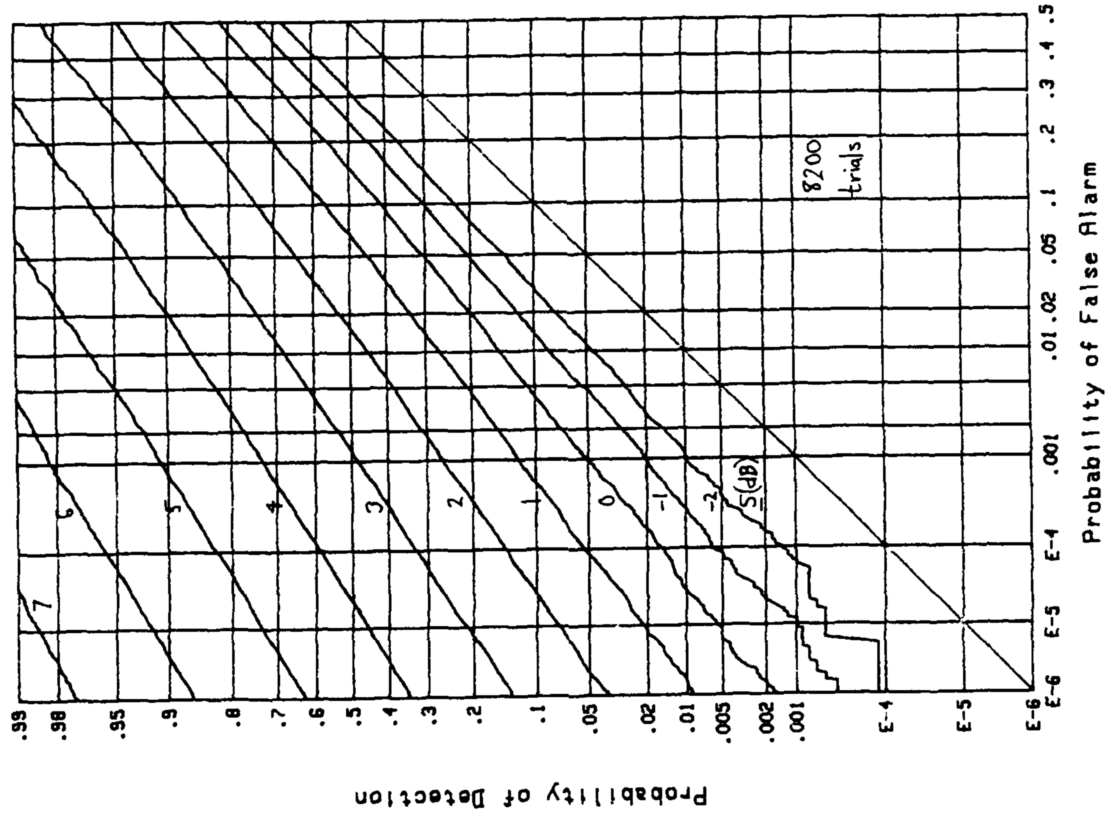

Figure E-63. ROC for SOML, M=32, M=4

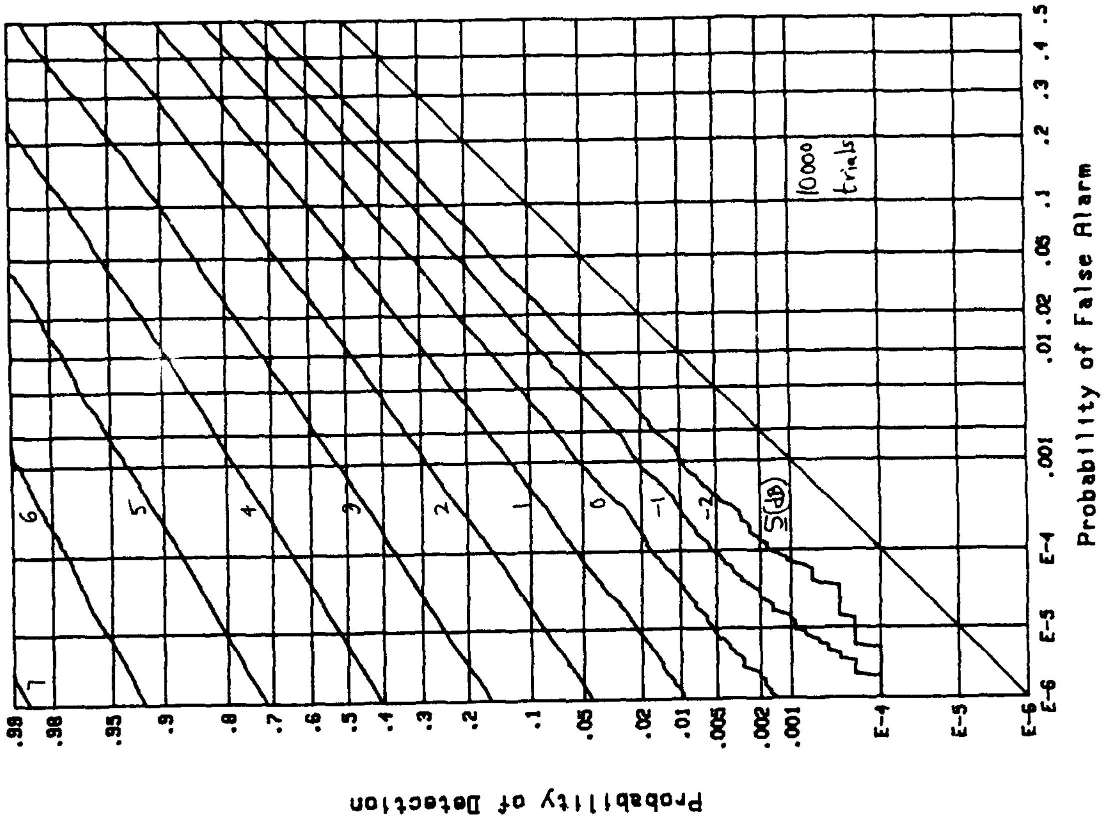

Figure E-64. ROC for SOML, M=32, M=8

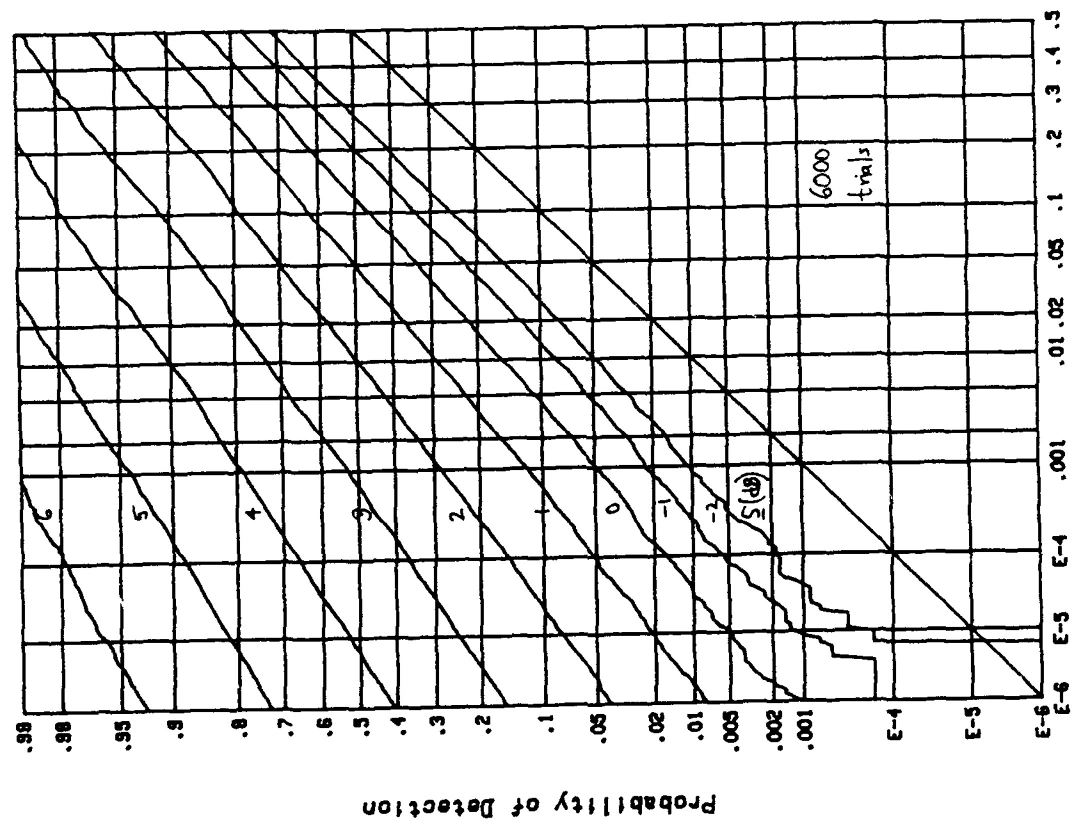

Figure E-65. ROC for SOML, M=32, M=16

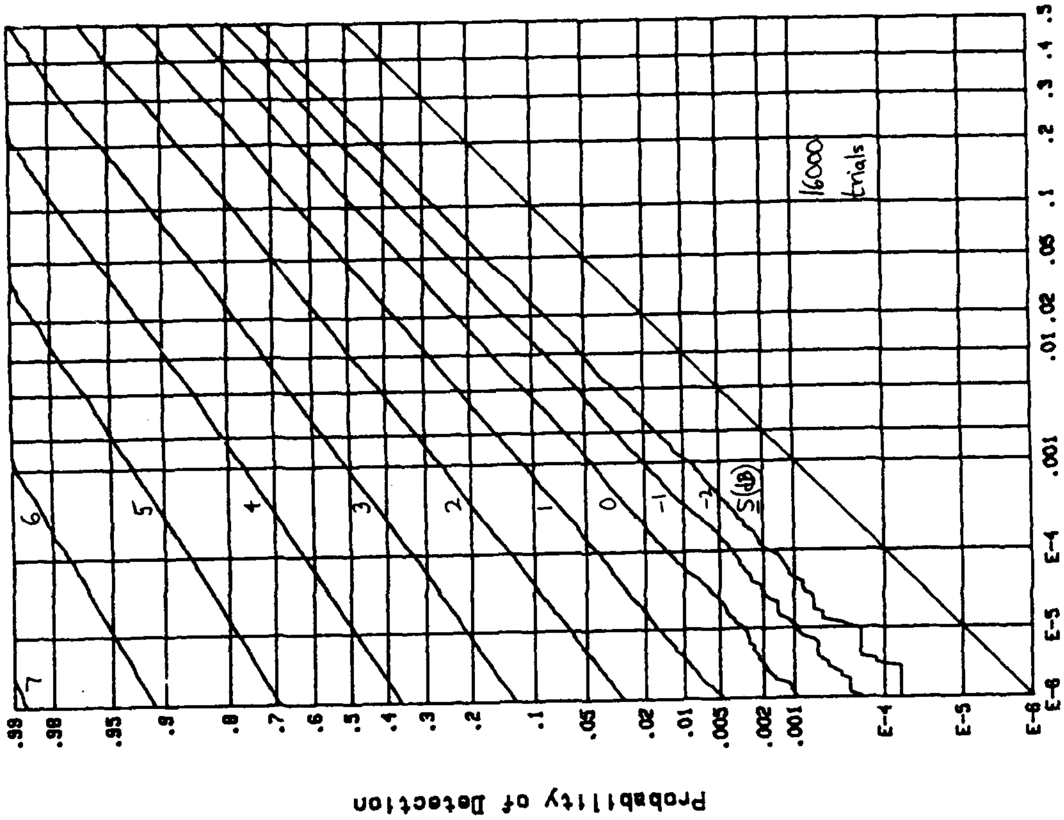

Figure E-66. ROC for SOML, M=32, M=32

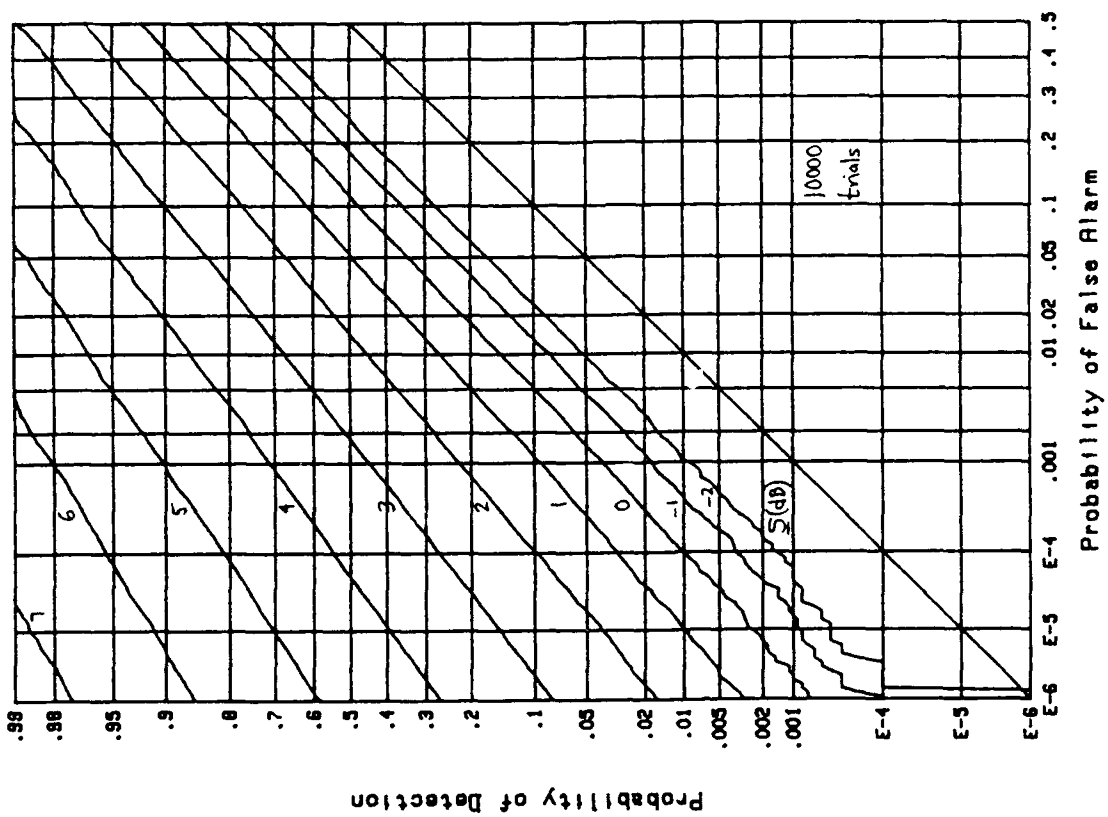

Figure E-67. ROC for SOML, M=32, M=64

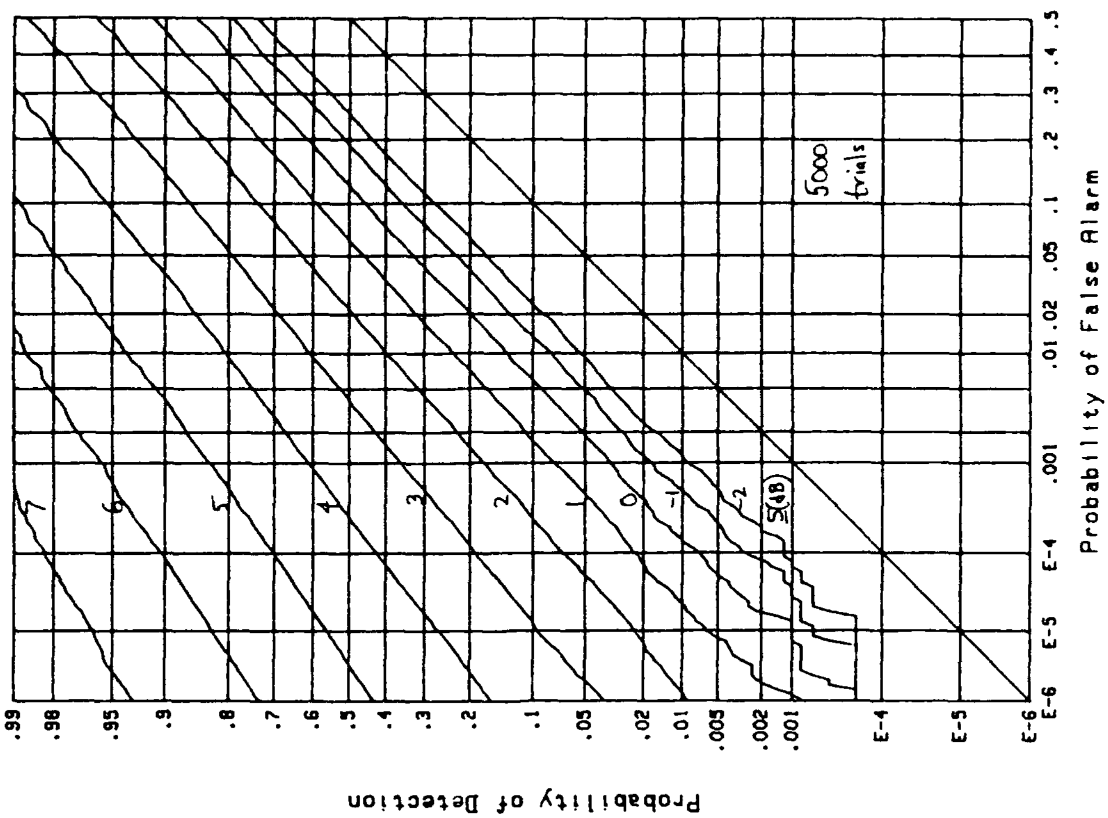

Figure E-68. ROC for SOML, M=32, M=128

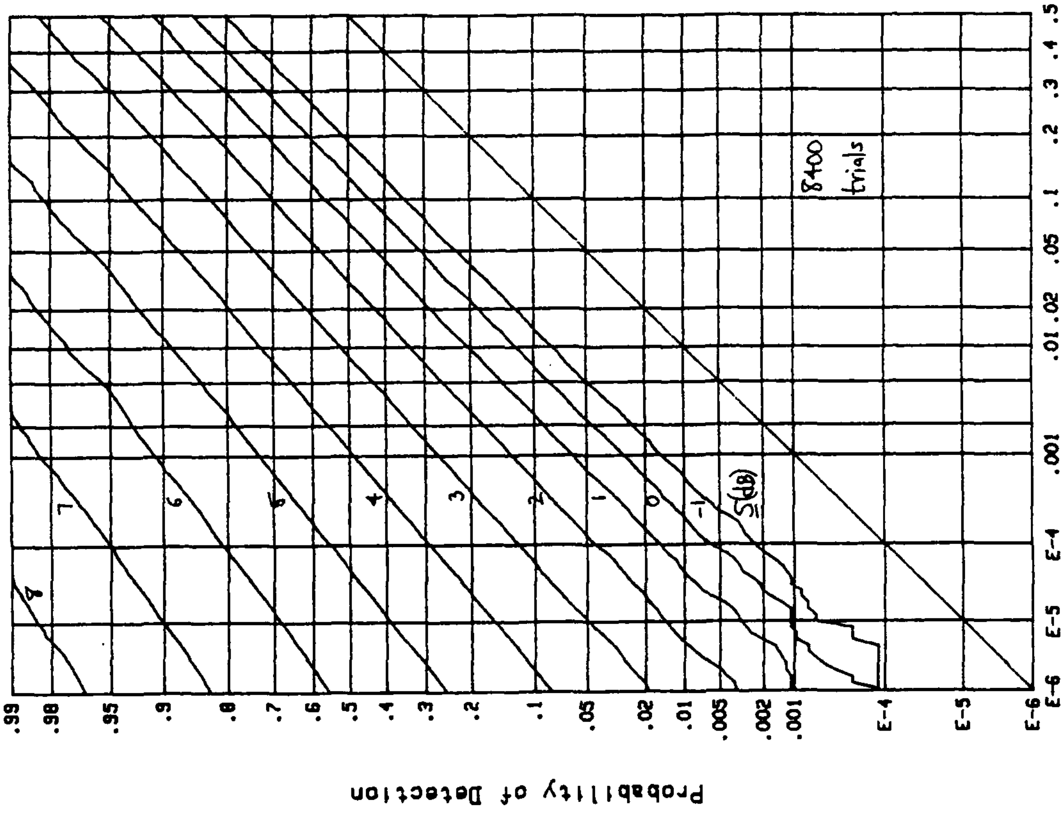

Figure E-69. ROC for SOML, $\underline{M}$=32, M=256

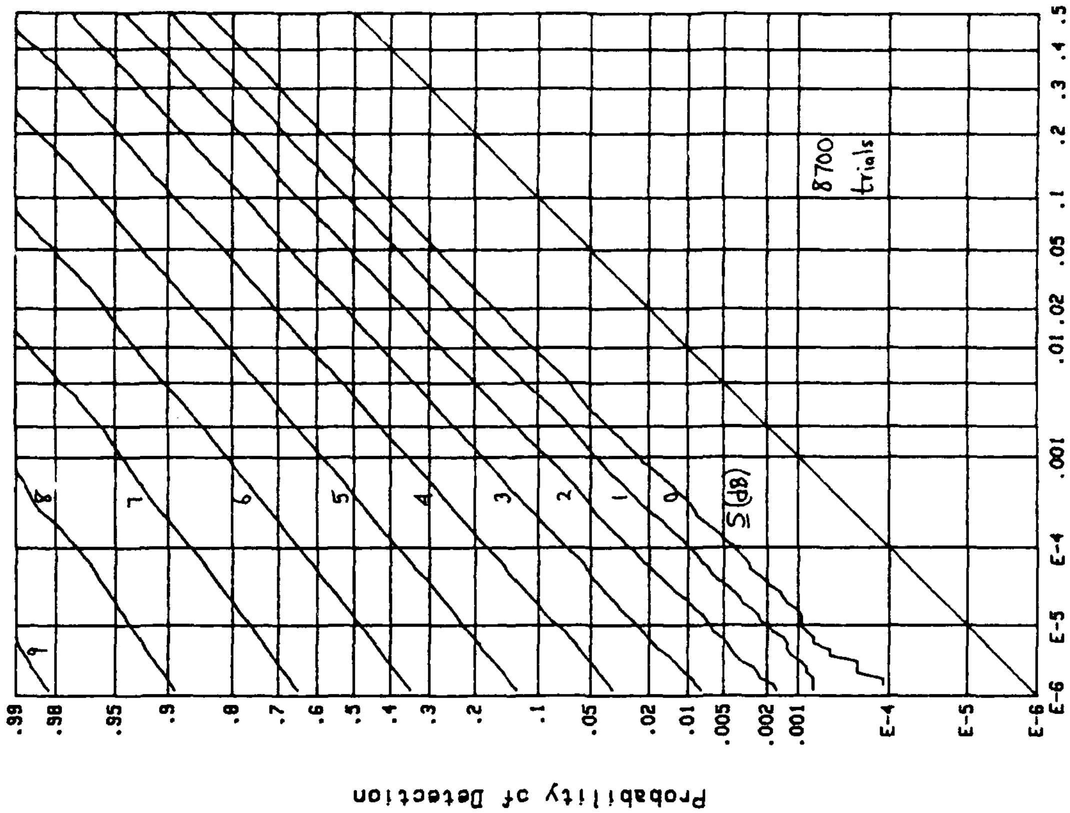

Figure E-70. ROC for SOML, $\underline{M}$=32, M=512

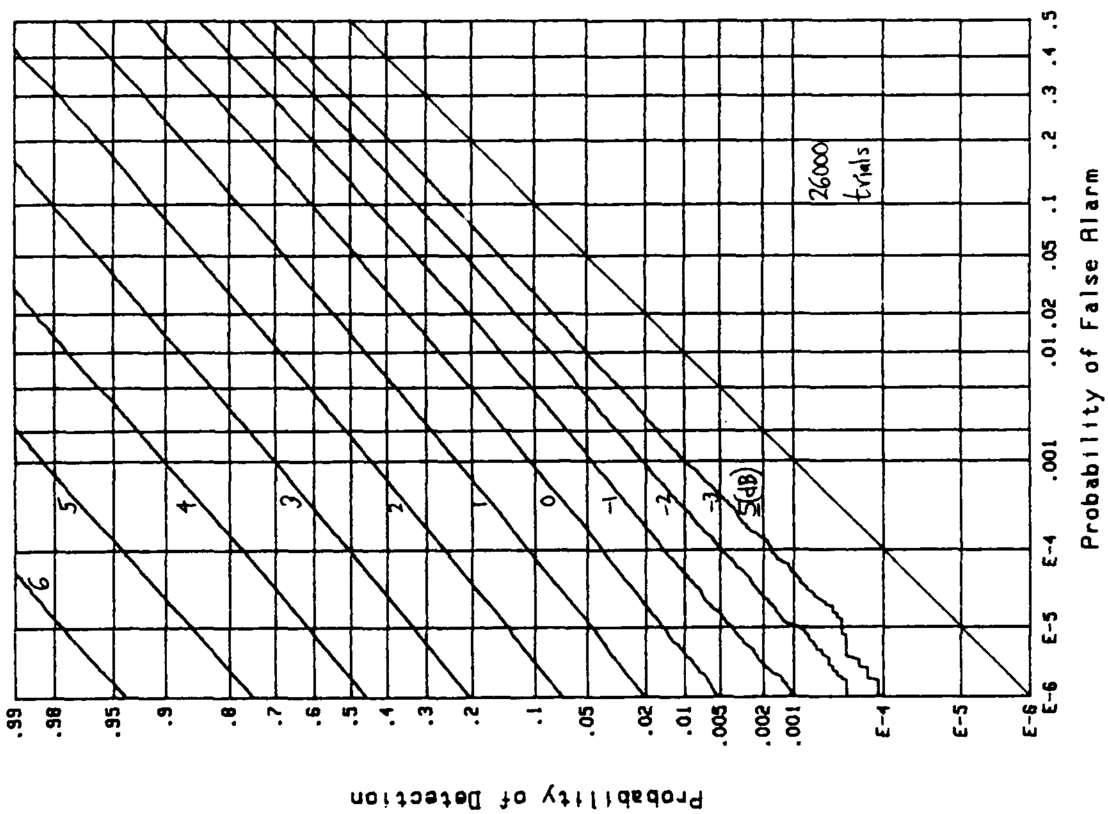

Figure E-71. ROC for SOML, M=64, M=2

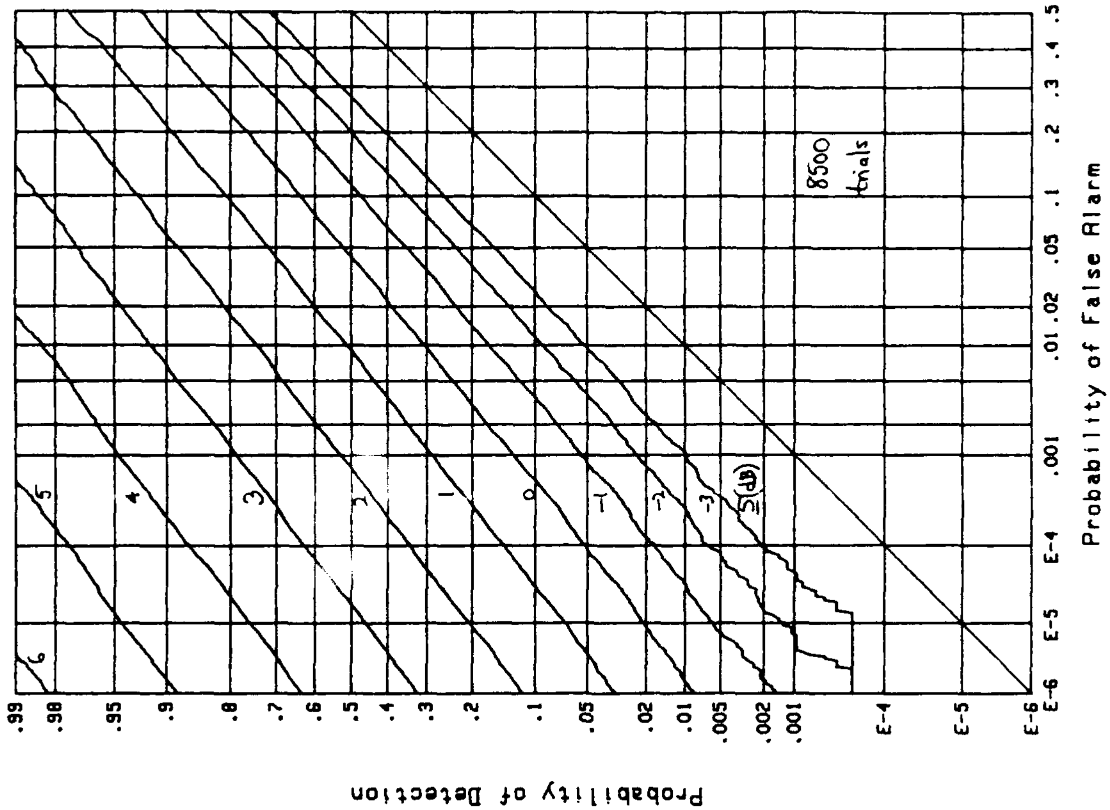

Figure E-72. ROC for SOML, M=64, M=3

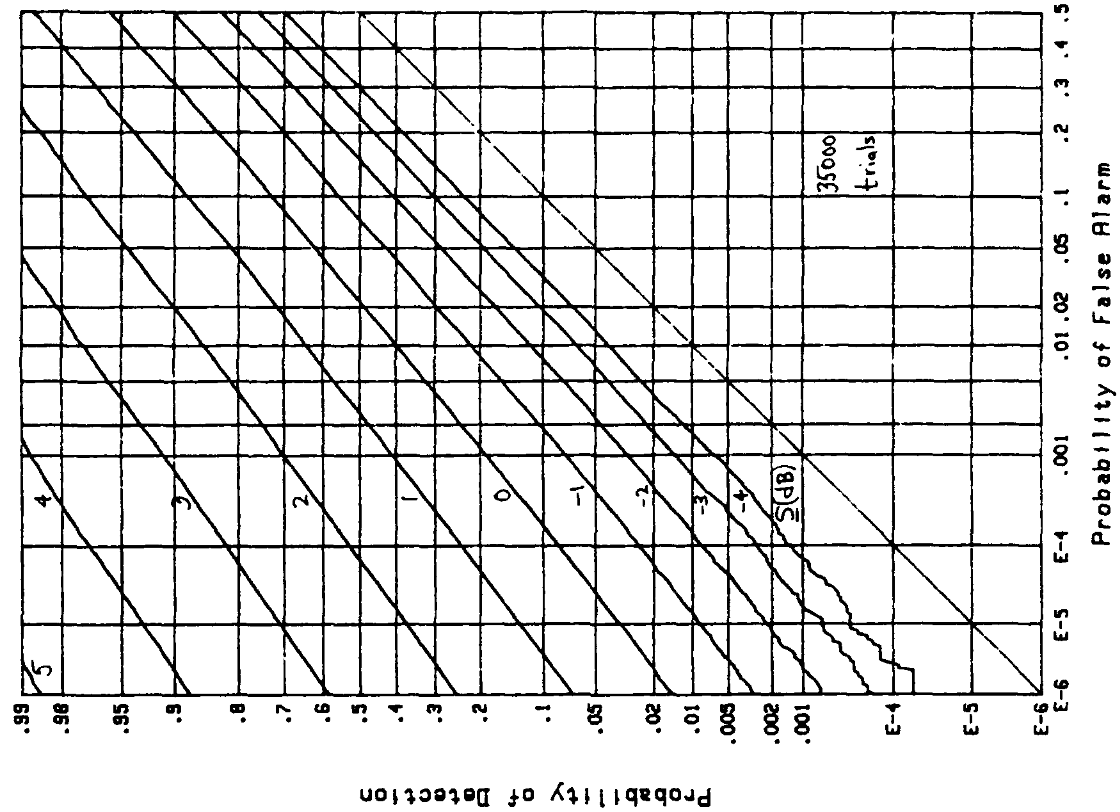

Figure E-74. ROC for SOML, $\underline{M}$=64, M=8

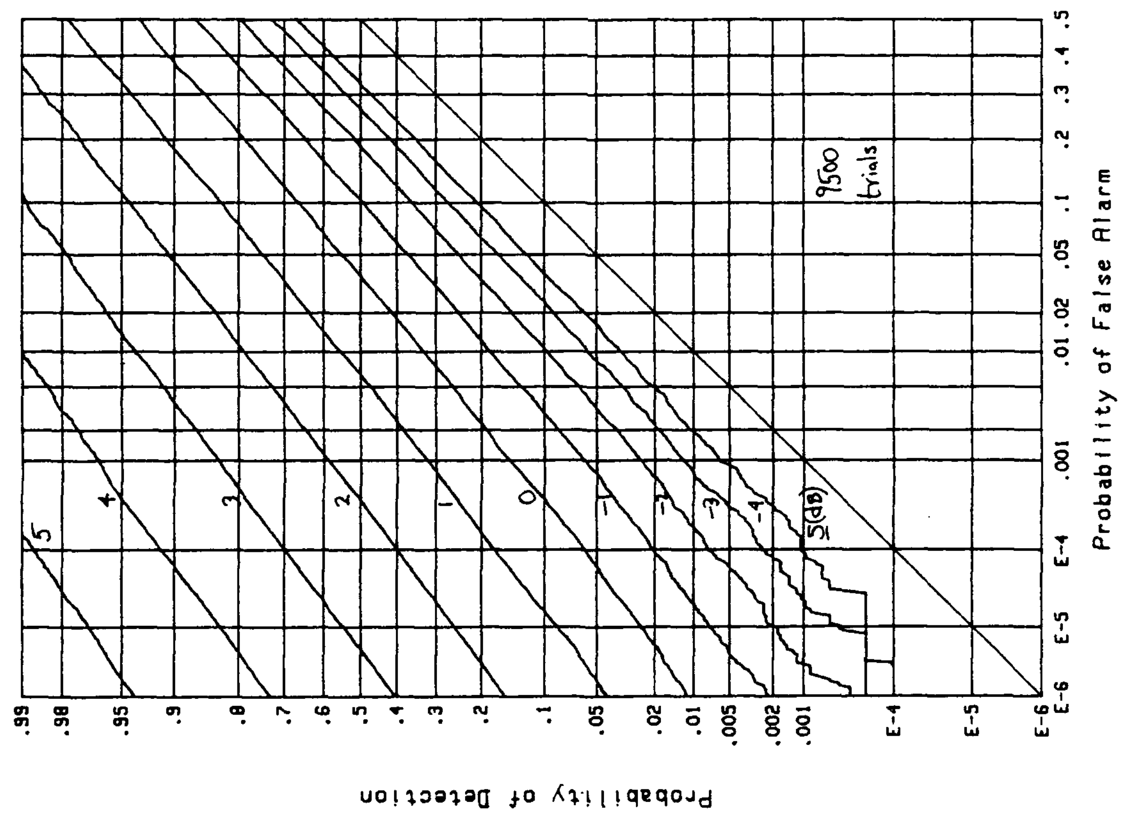

Figure E-73. ROC for SOML, $\underline{M}$=64, M=4

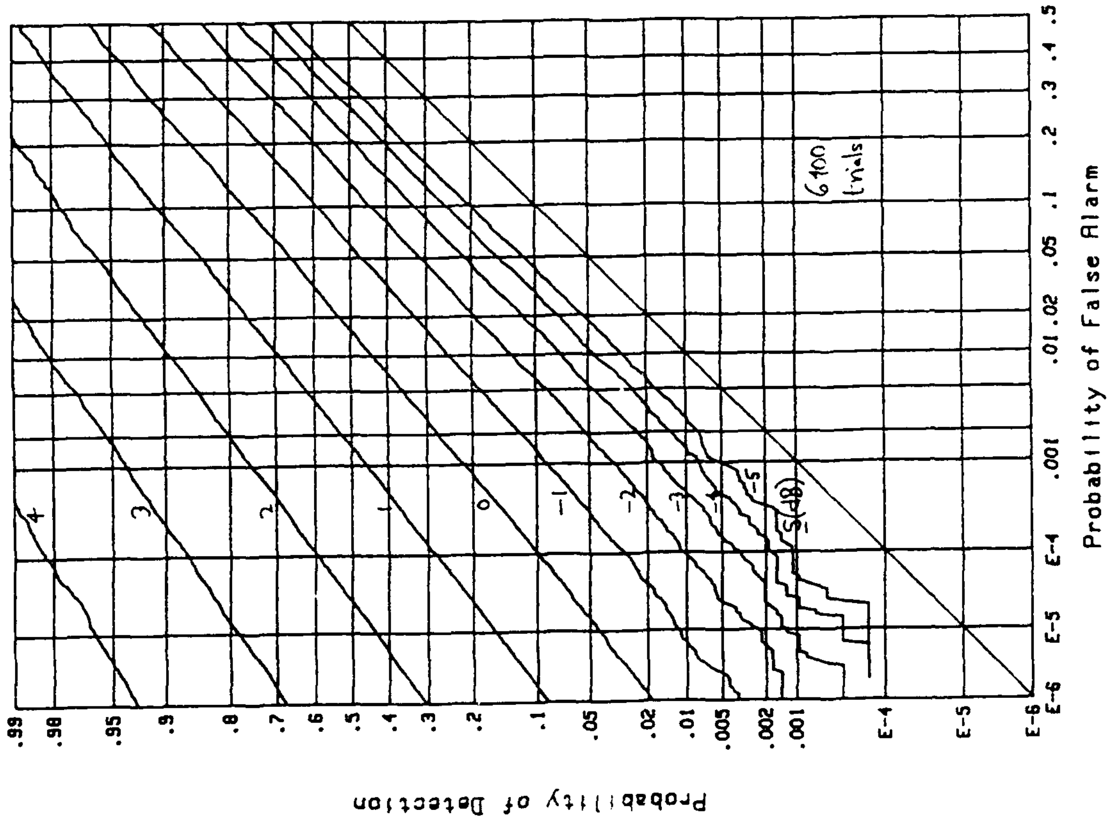

Figure E-75. ROC for SOML, $\underline{M}$=64, M=16

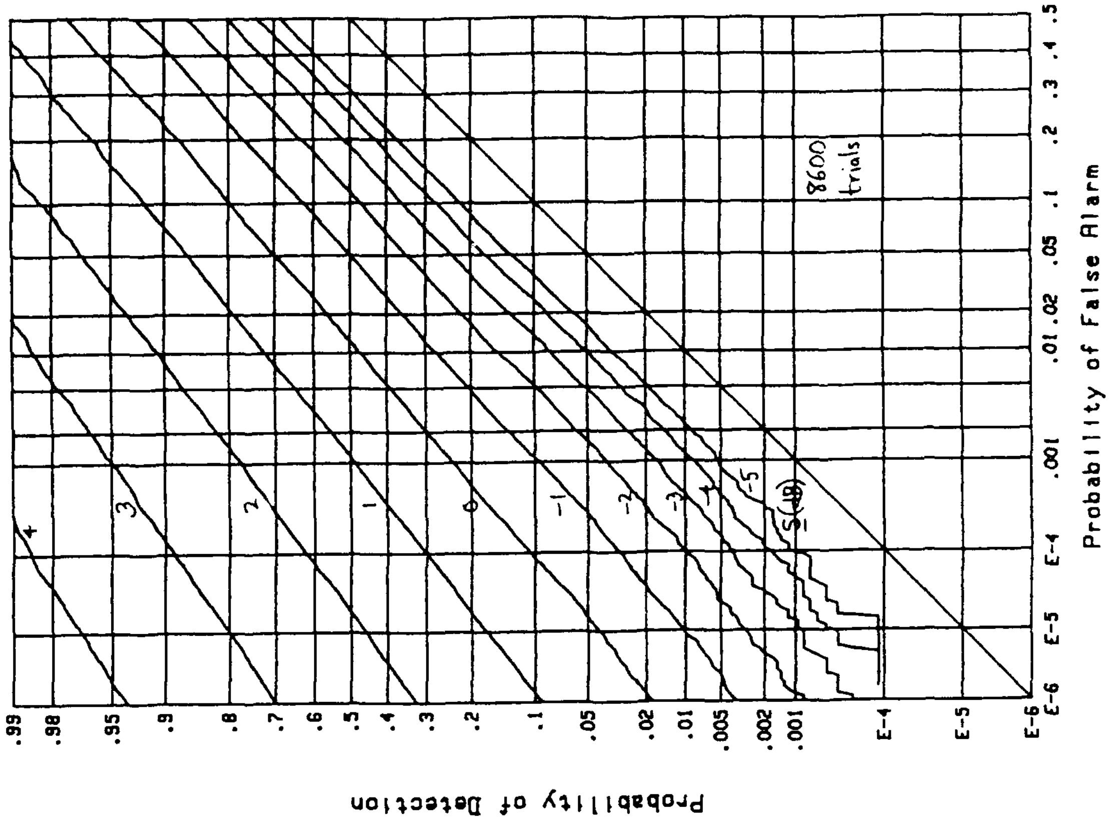

Figure E-76. ROC for SOML, $\underline{M}$=64, M=32

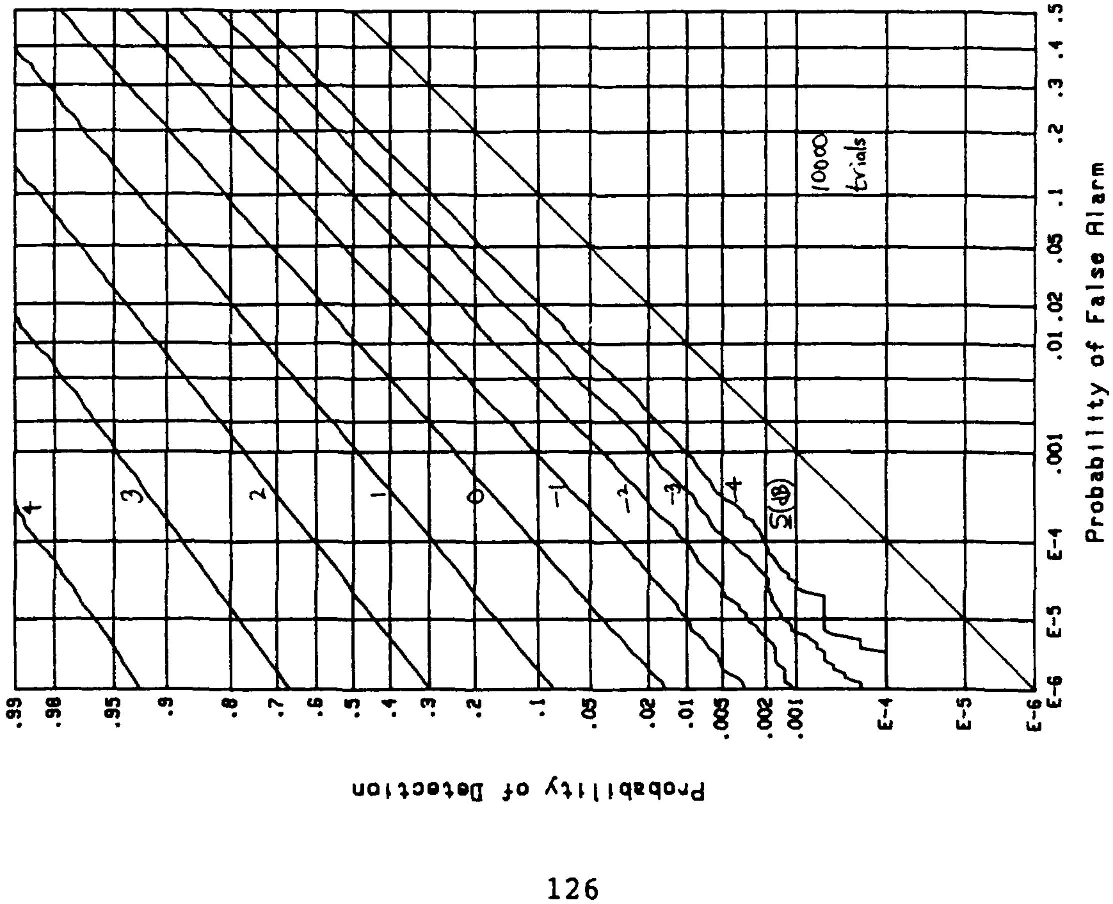

Figure E-77. ROC for SOML, $\underline{M}$=64, M=64

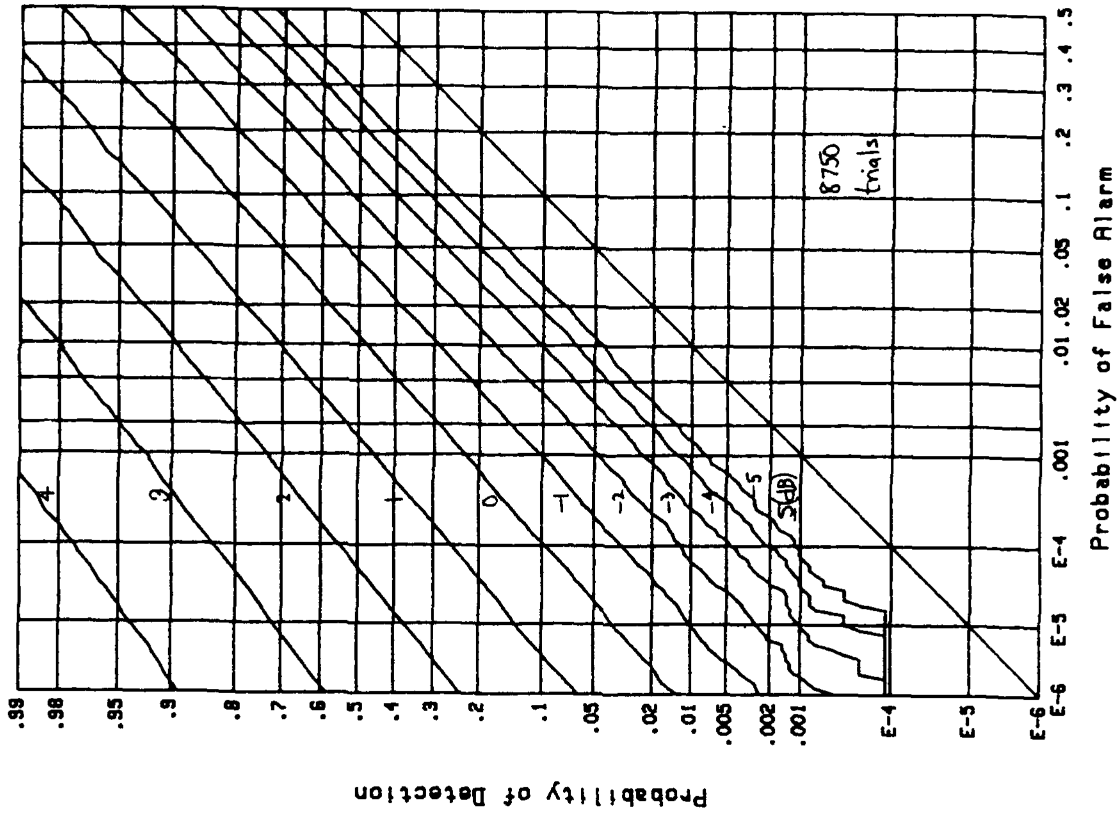

Figure E-78. ROC for SOML, $\underline{M}$=64, M=128

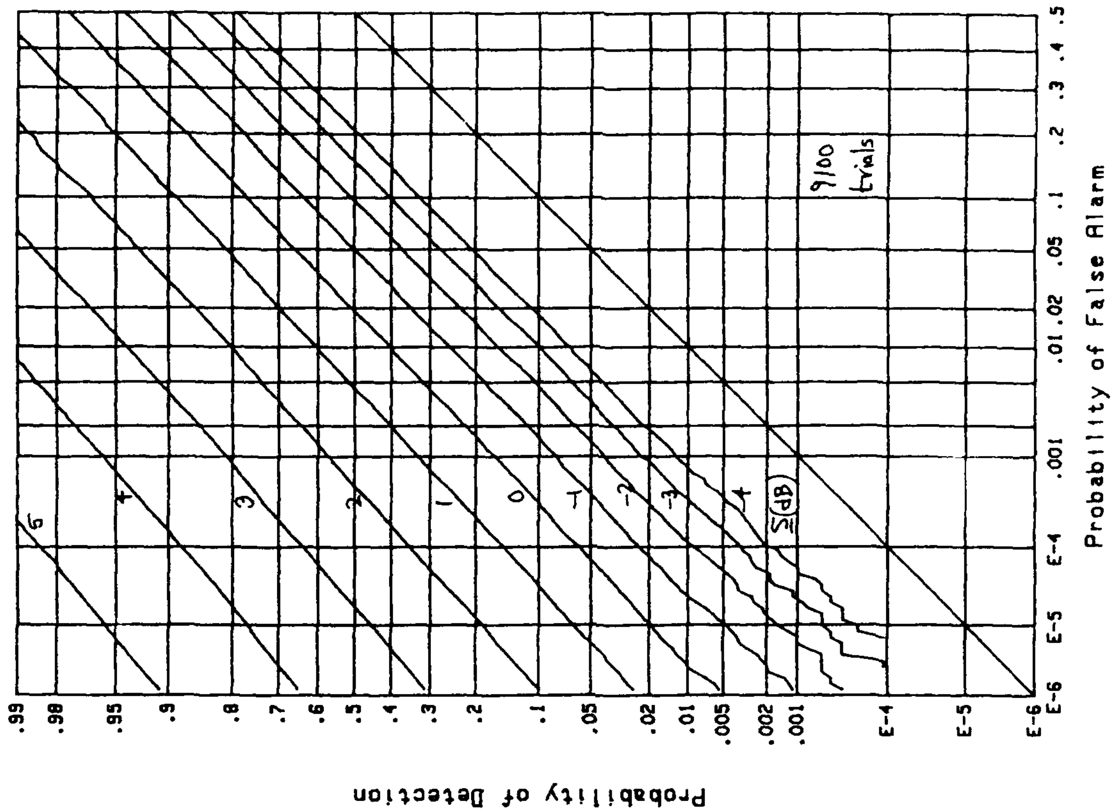

Figure E-80. ROC for SOML, M=64, M=512

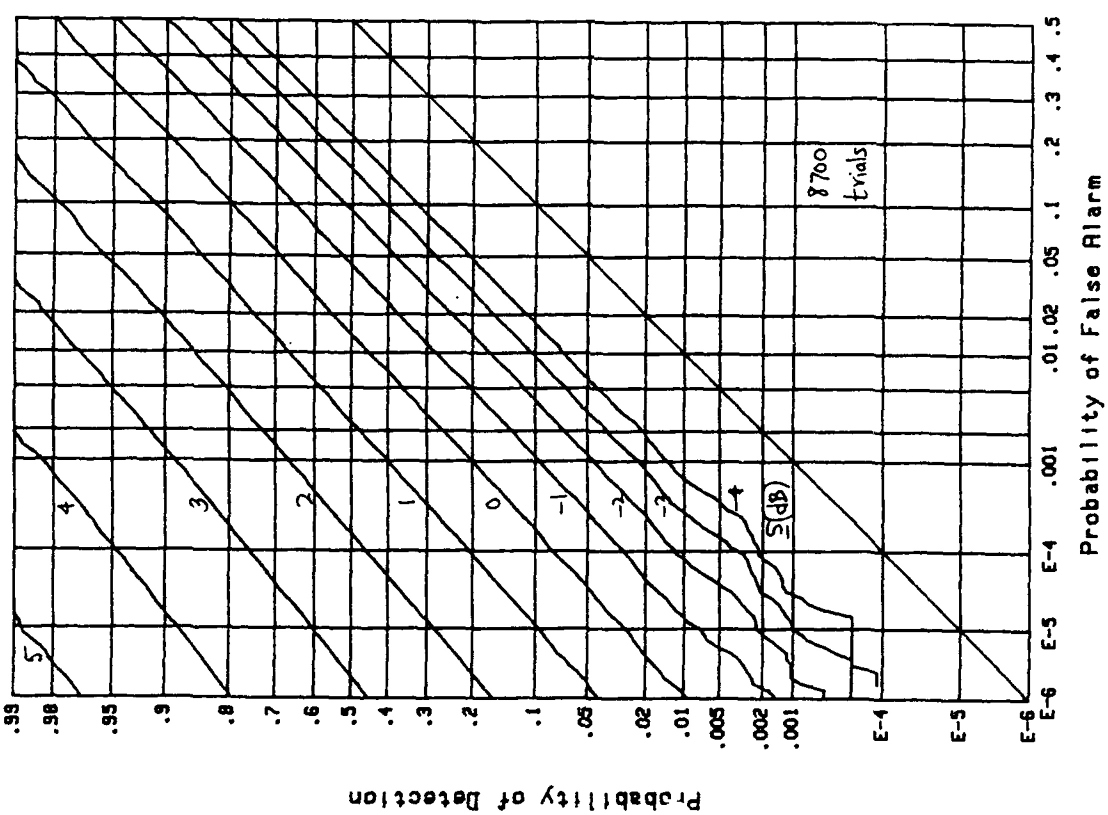

Figure E-79. ROC for SOML, M=64, M=256

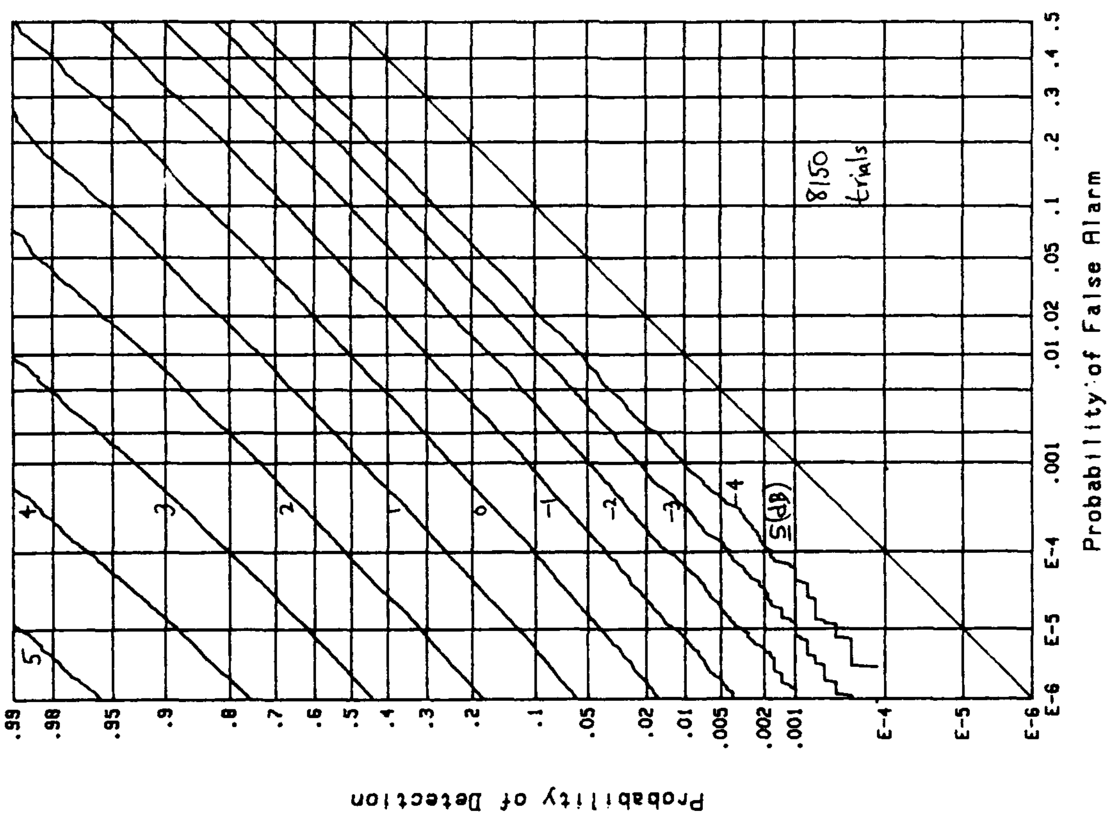

Figure E-81. ROC for SOML, M̲=128, M=2

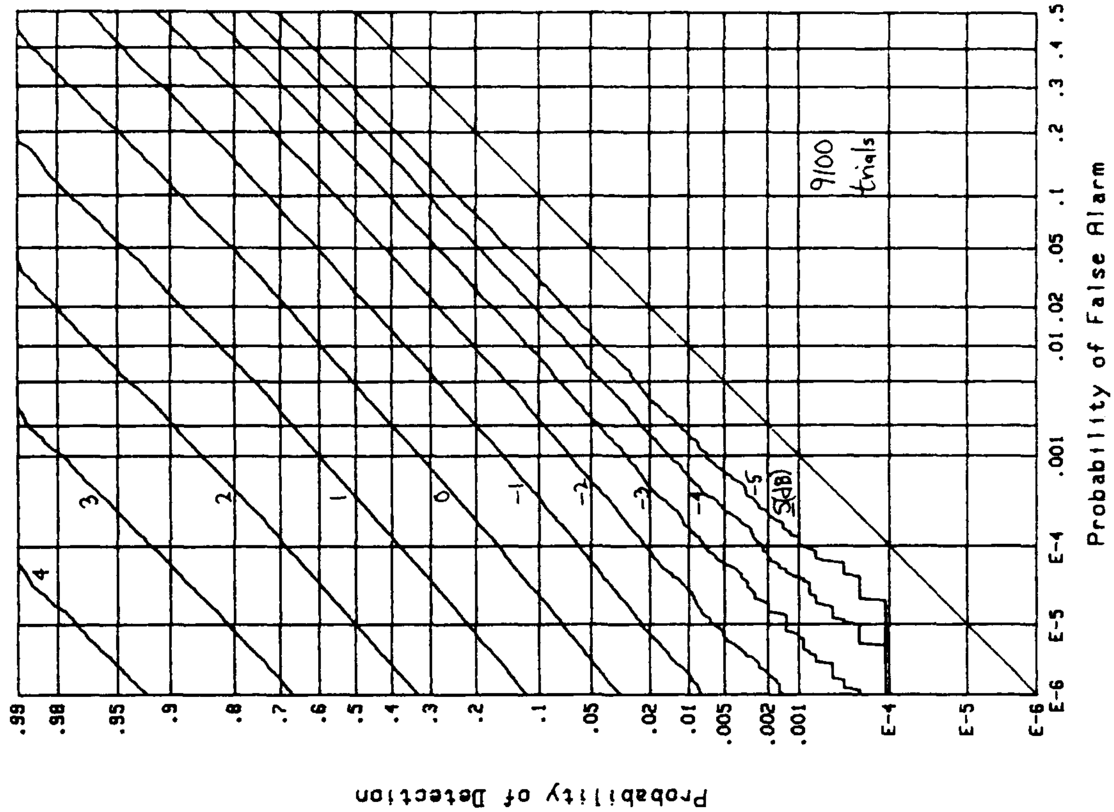

Figure E-82. ROC for SOML, M̲=128, M=3

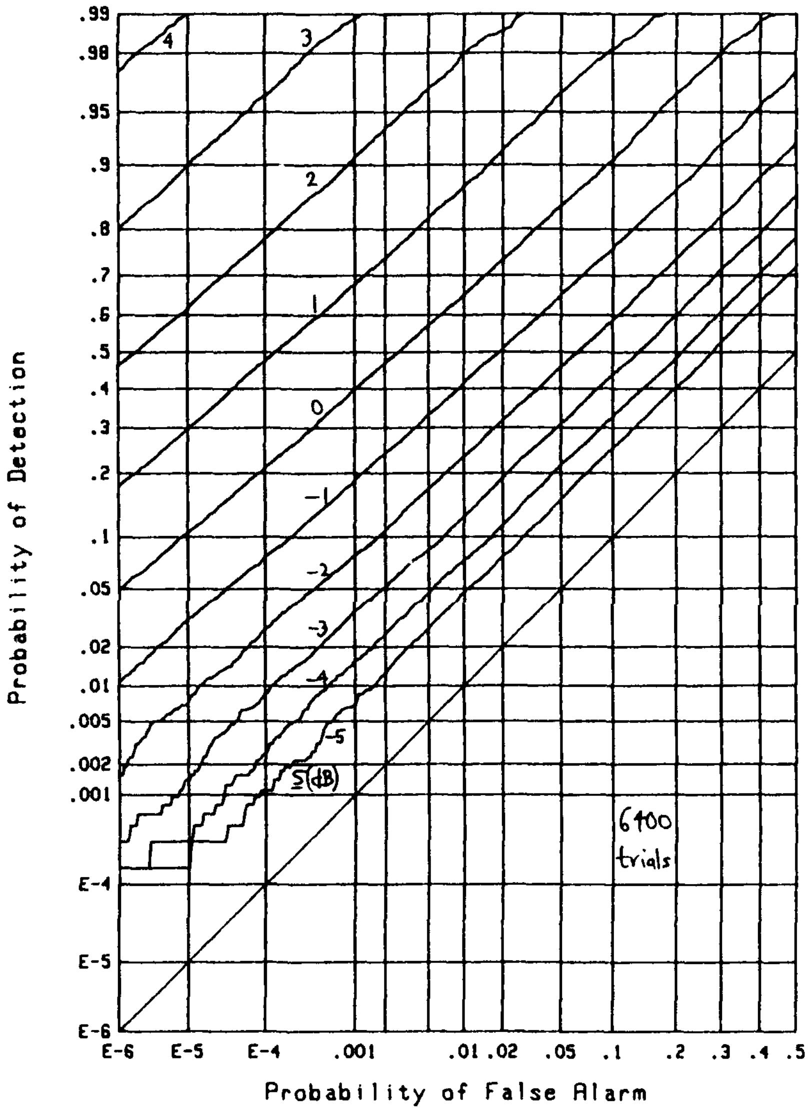

Figure E-83. ROC for SOML, M=128, M=4

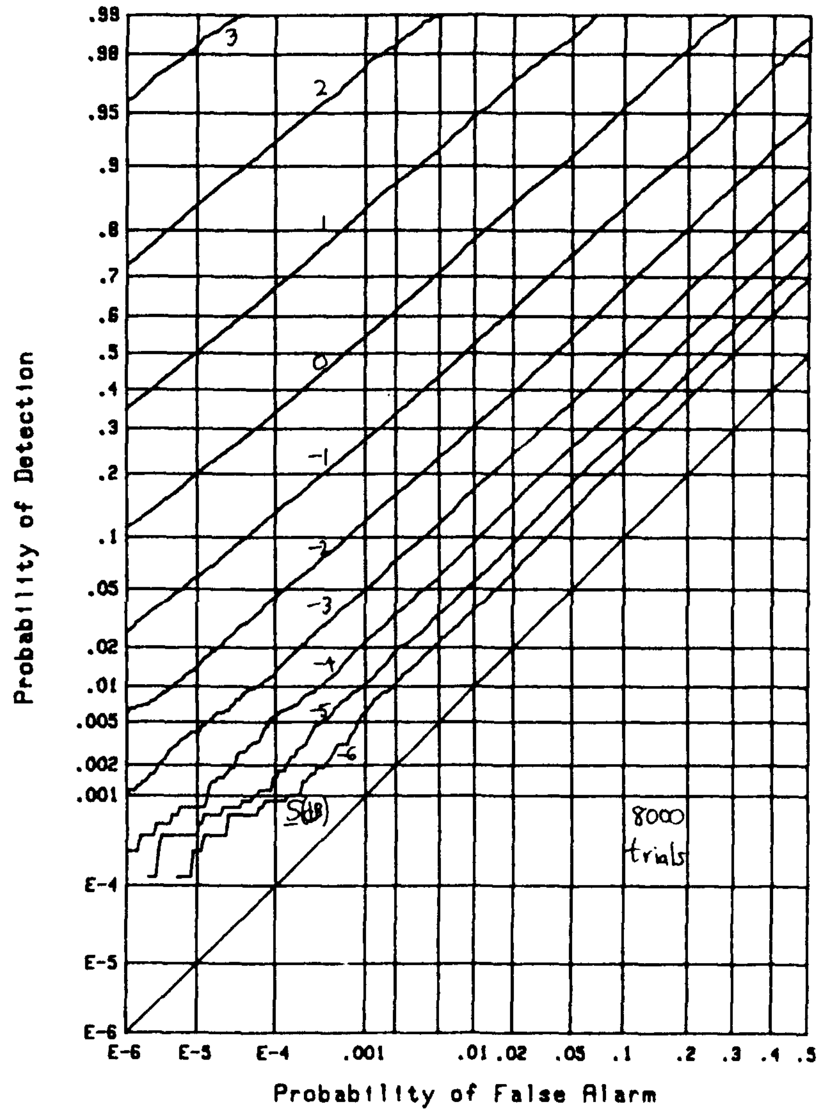

Figure E-84. ROC for SOML, M=128, M=8

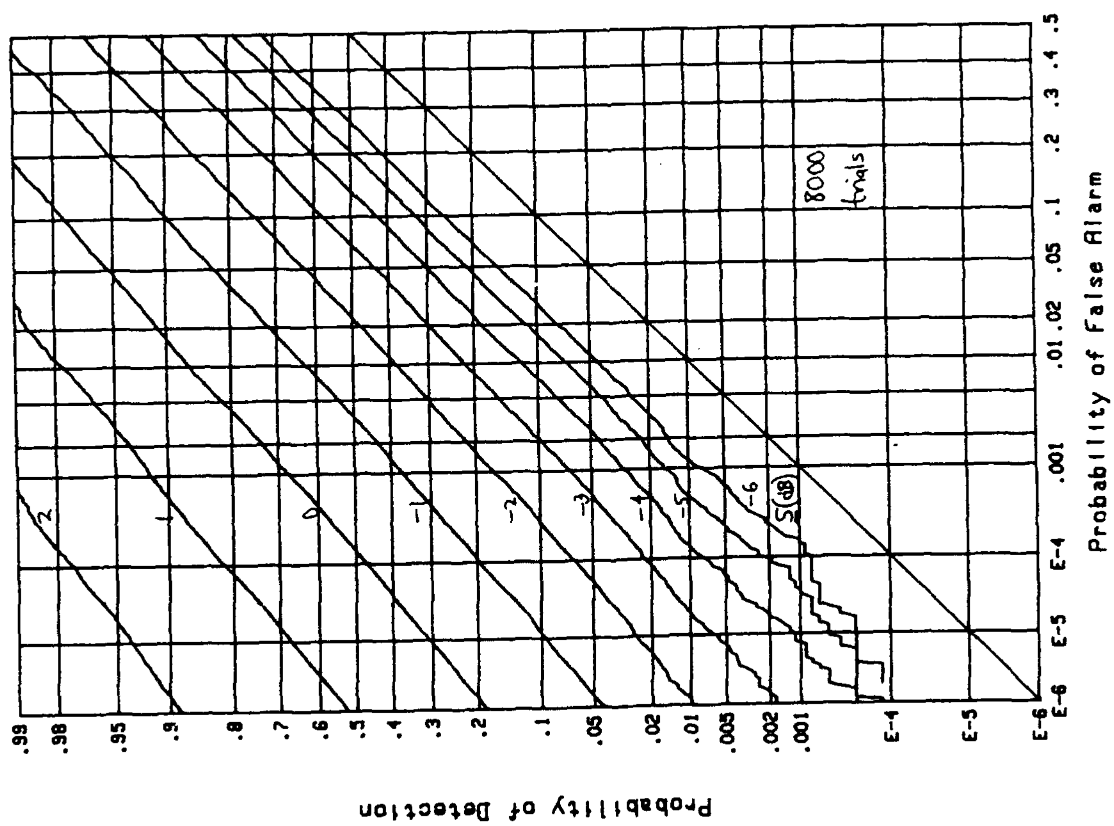

Figure E-85. ROC for SOML, M=128, M=16

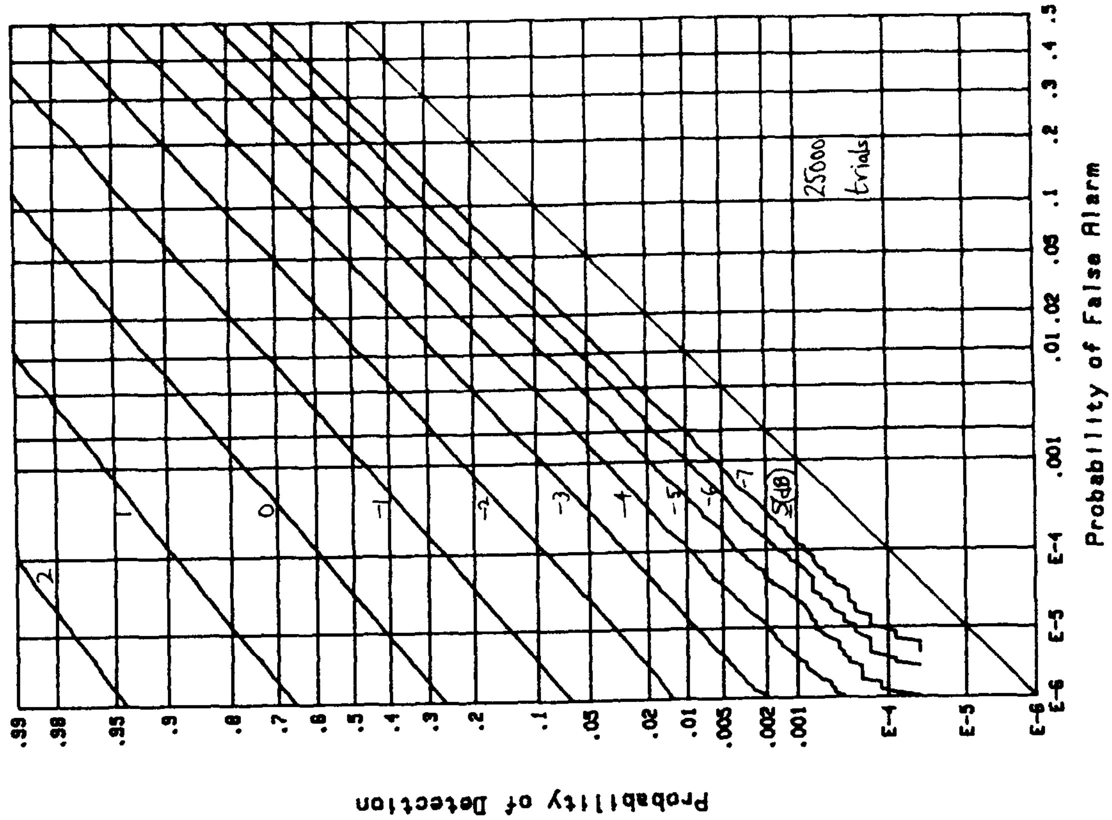

Figure E-86. ROC for SOML, M=128, M=32

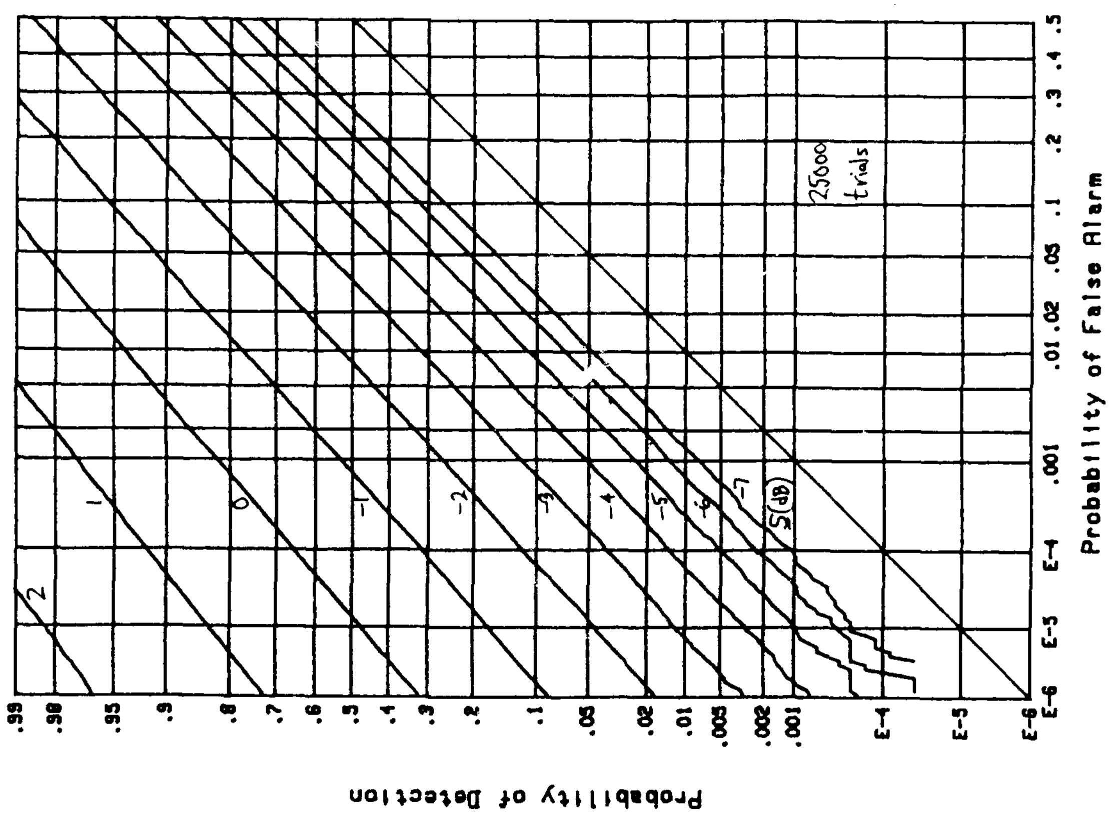

Figure E-87. ROC for SOML, M=128, M=64

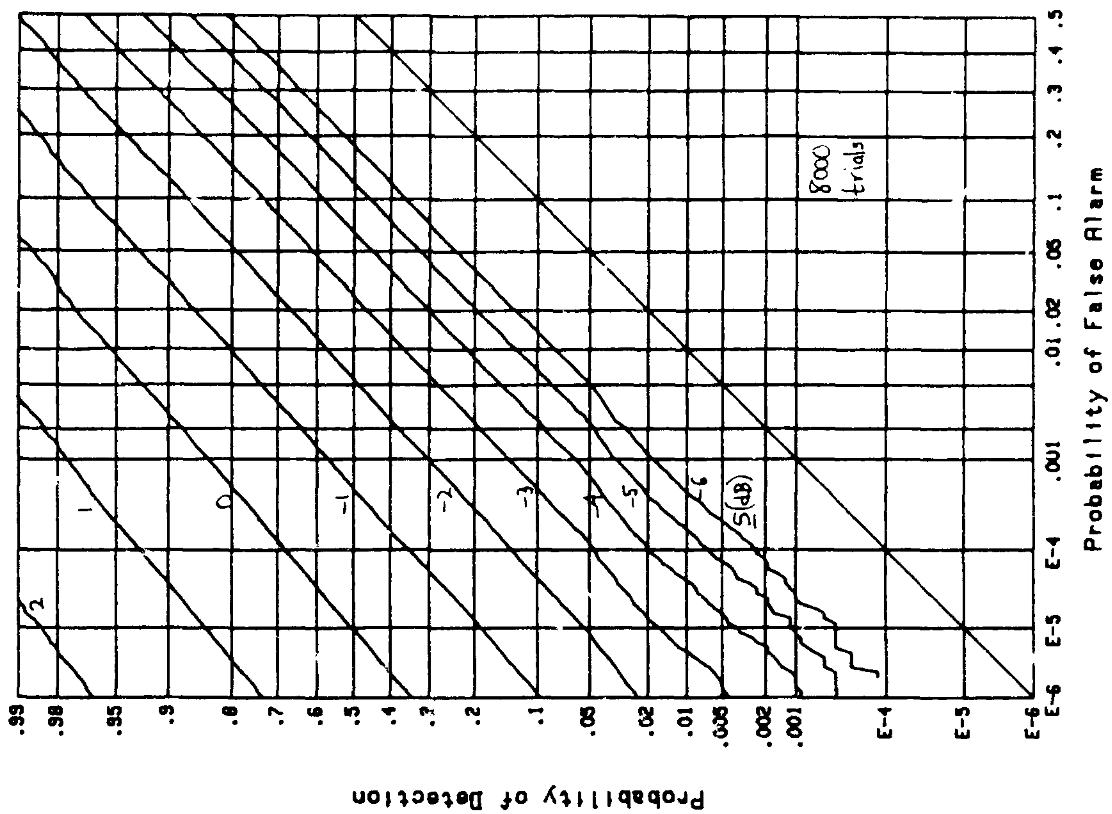

Figure E-88. ROC for SOML, M=128, M=128

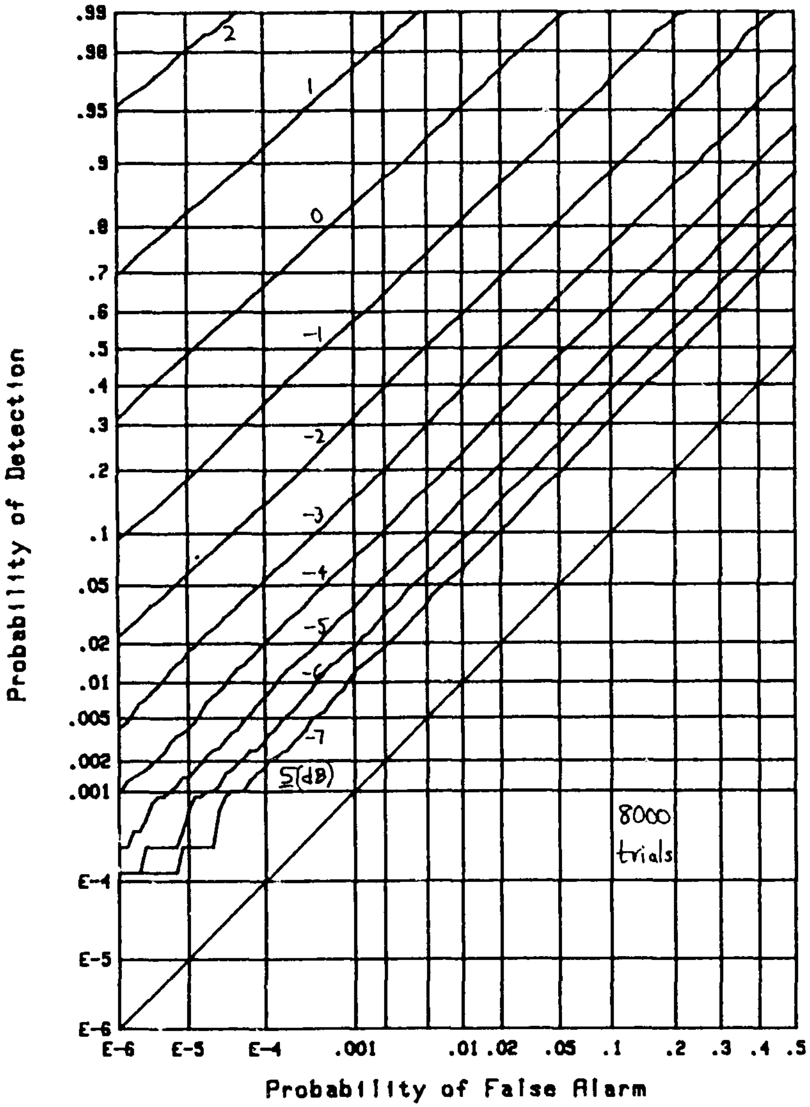

Figure E-89. ROC for SOML, M=128, M=256

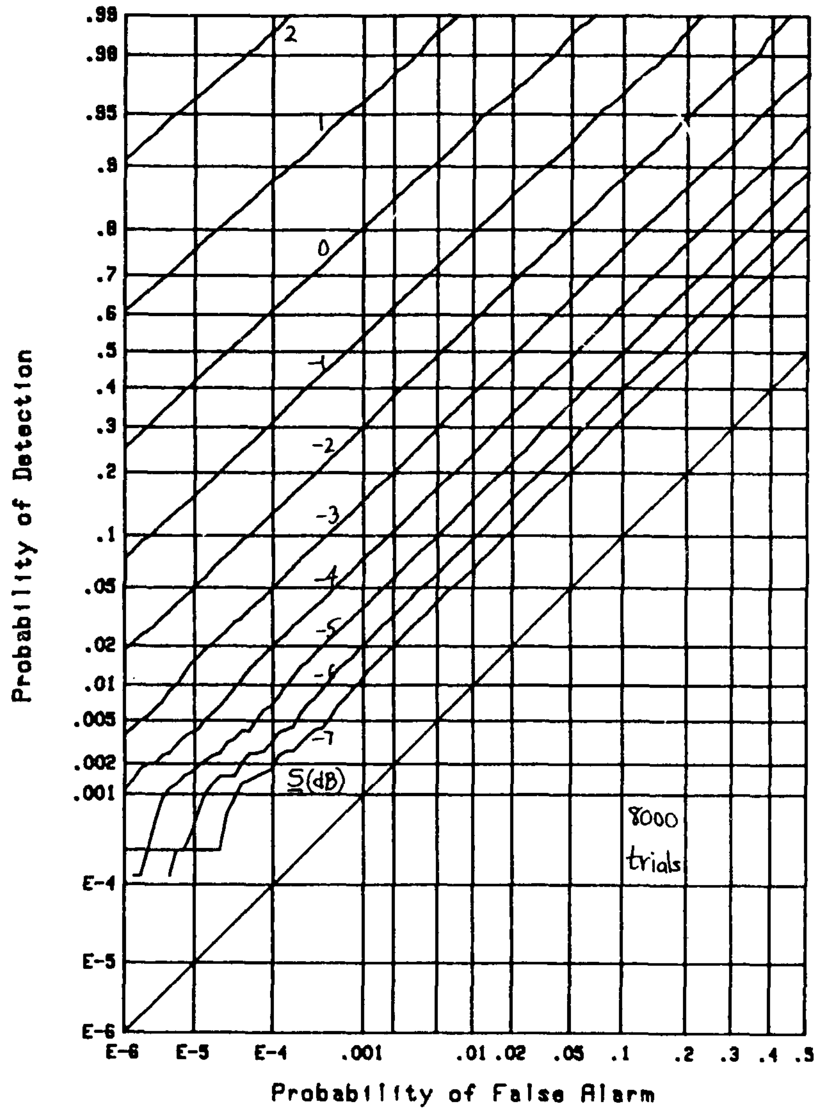

Figure E-90. ROC for SOML, M=128, M=512

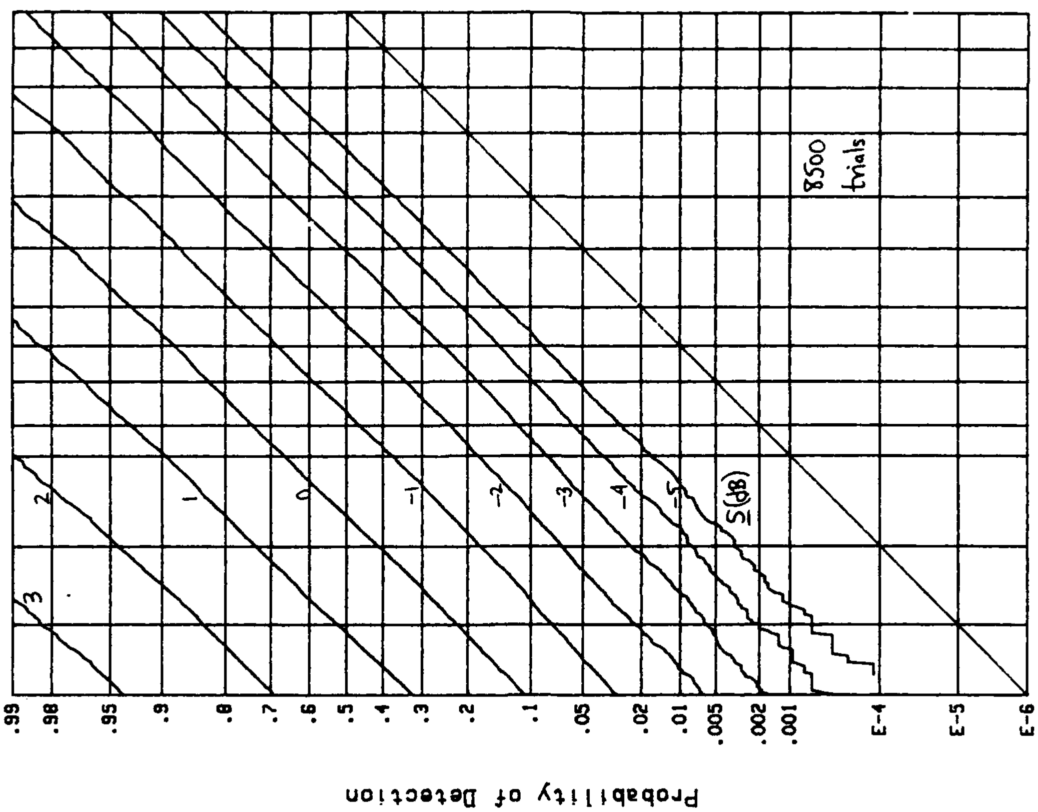

Figure E-92. ROC for SOML, M=256, M=3

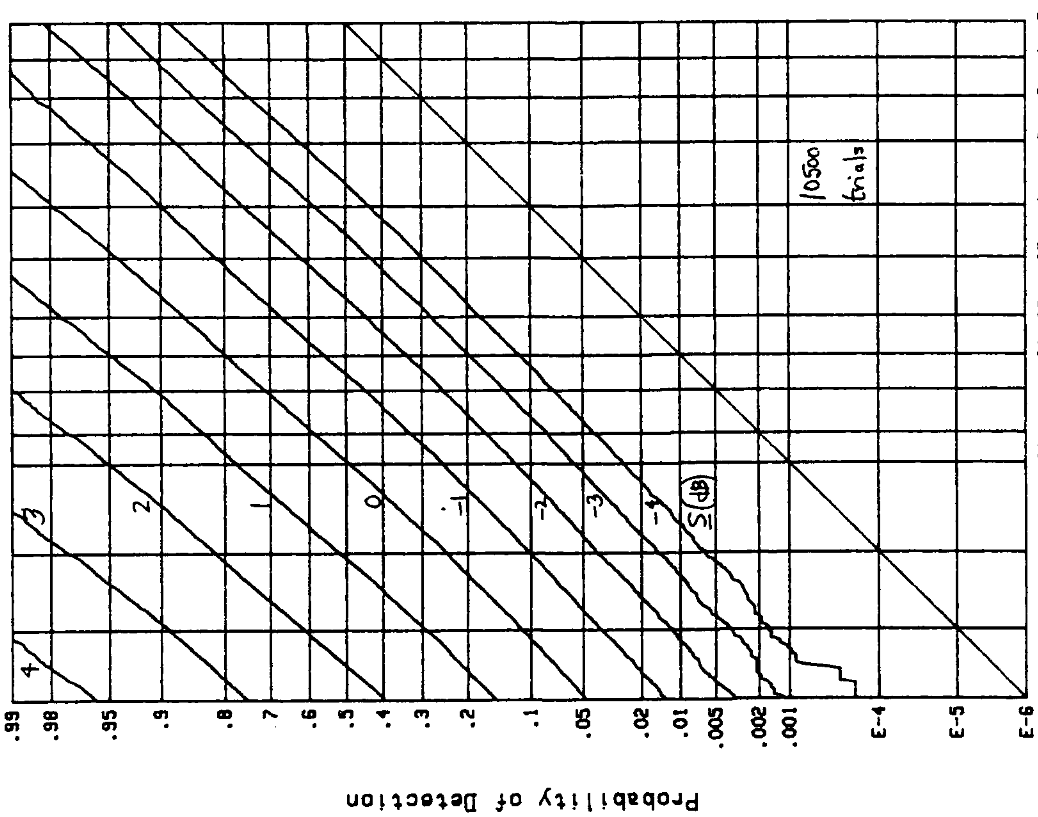

Figure E-91. ROC for SOML, M=256, M=2

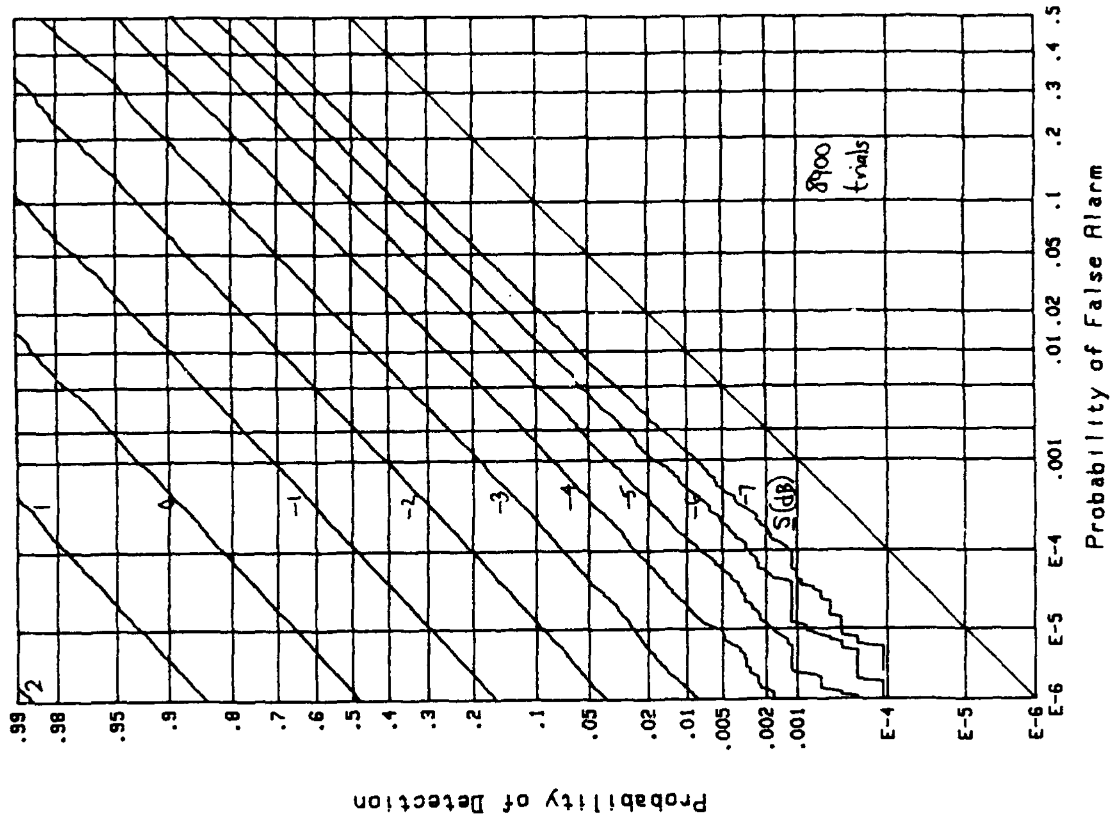

Figure E-94. ROC for SOML, M̲=256, M=8

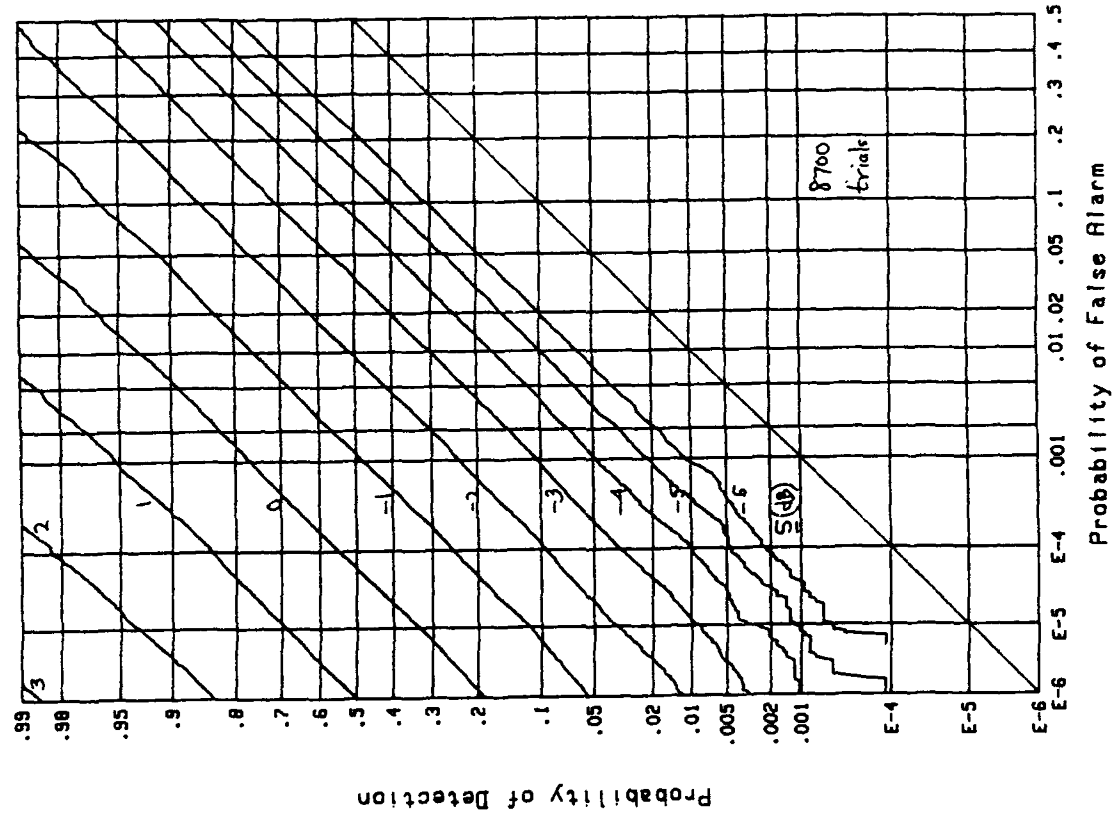

Figure E-93. ROC for SOML, M̲=256, M=4

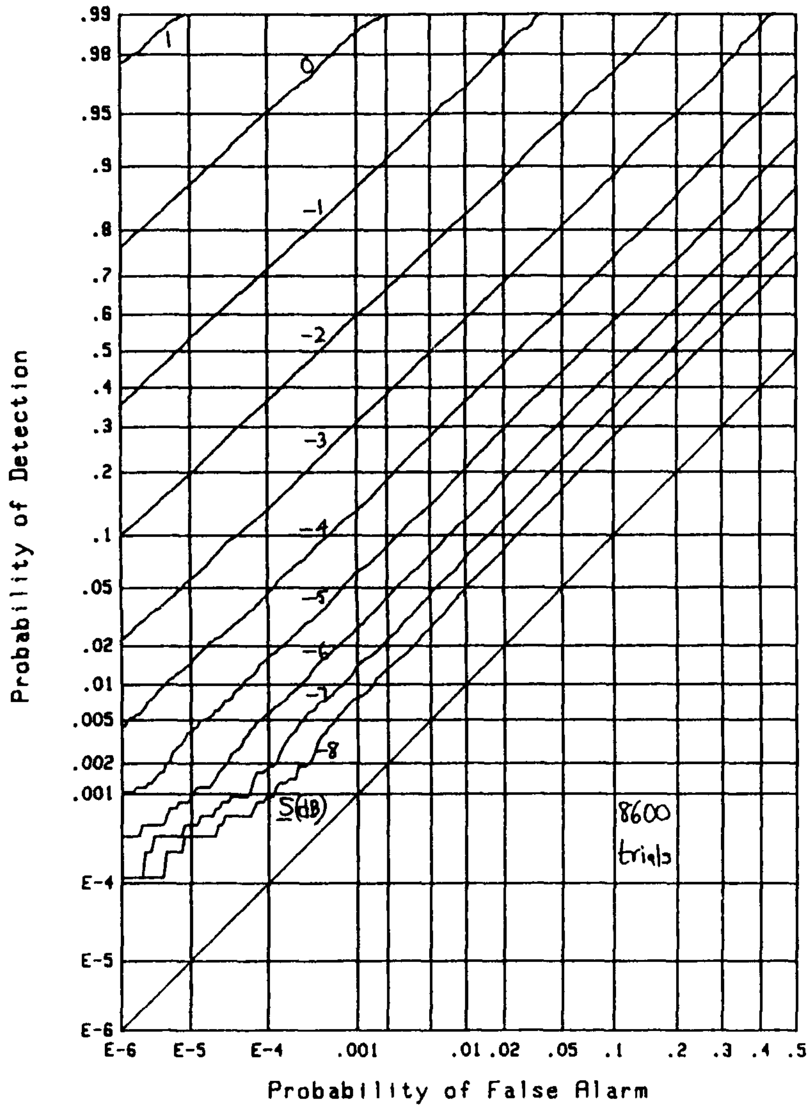

Figure E-95. ROC for SOML, $\underline{M}$=256, M=16

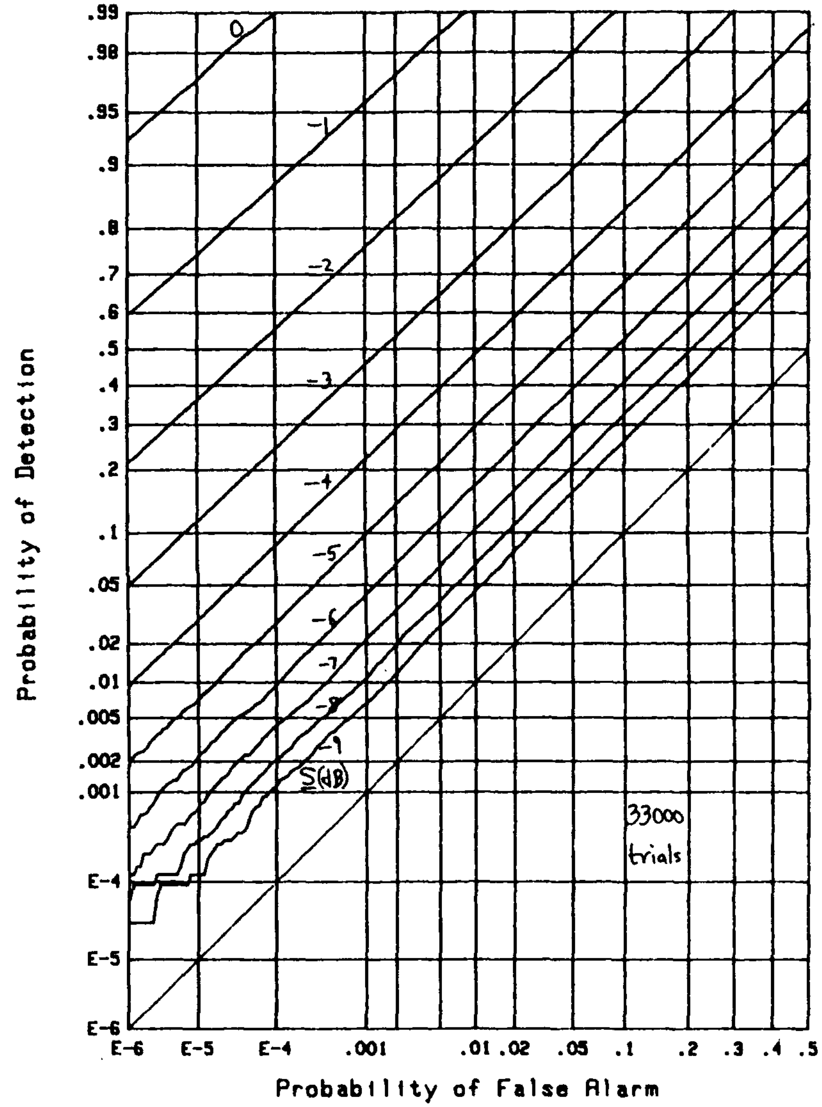

Figure E-96. ROC for SOML, $\underline{M}$=256, M=32

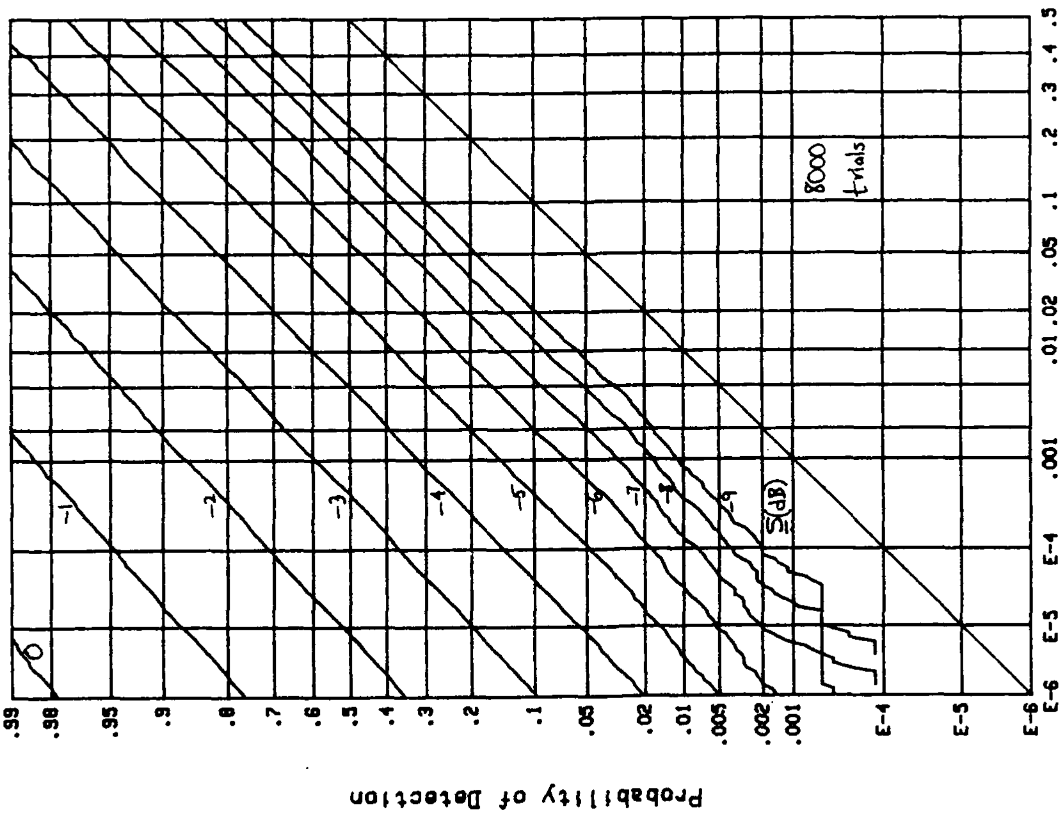

Figure E-97. ROC for SOML, M=256, M=64

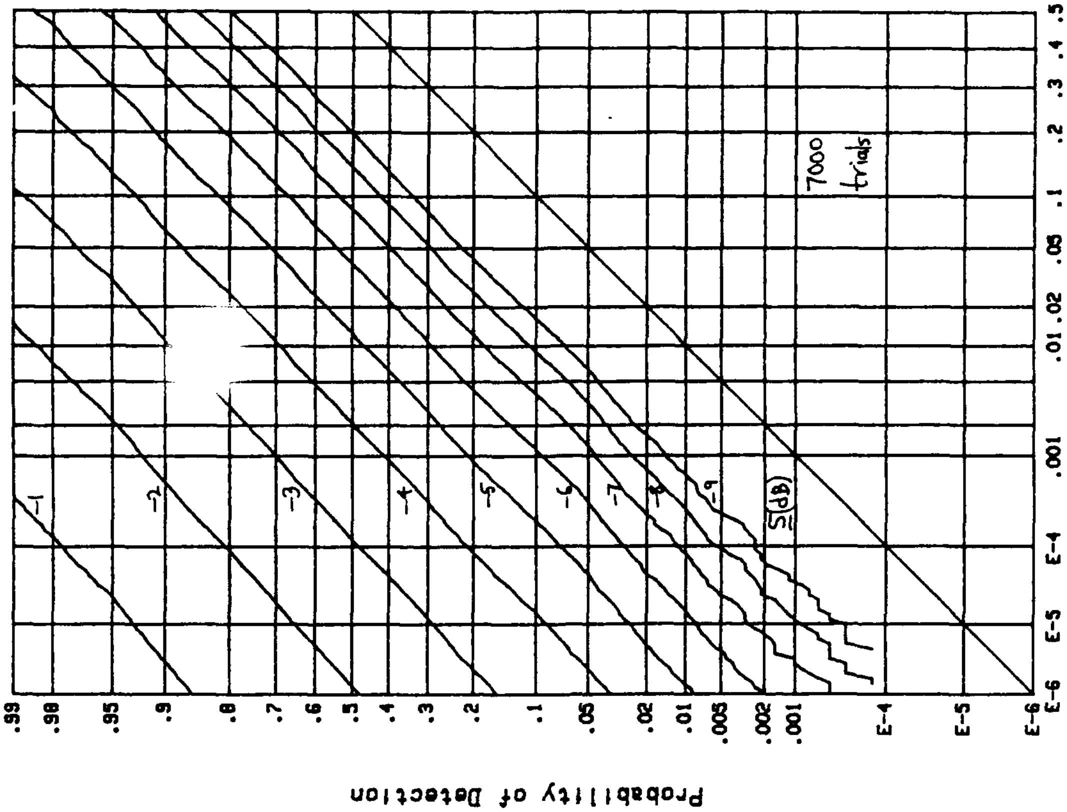

Figure E-98. ROC for SOML, M=256, M=128

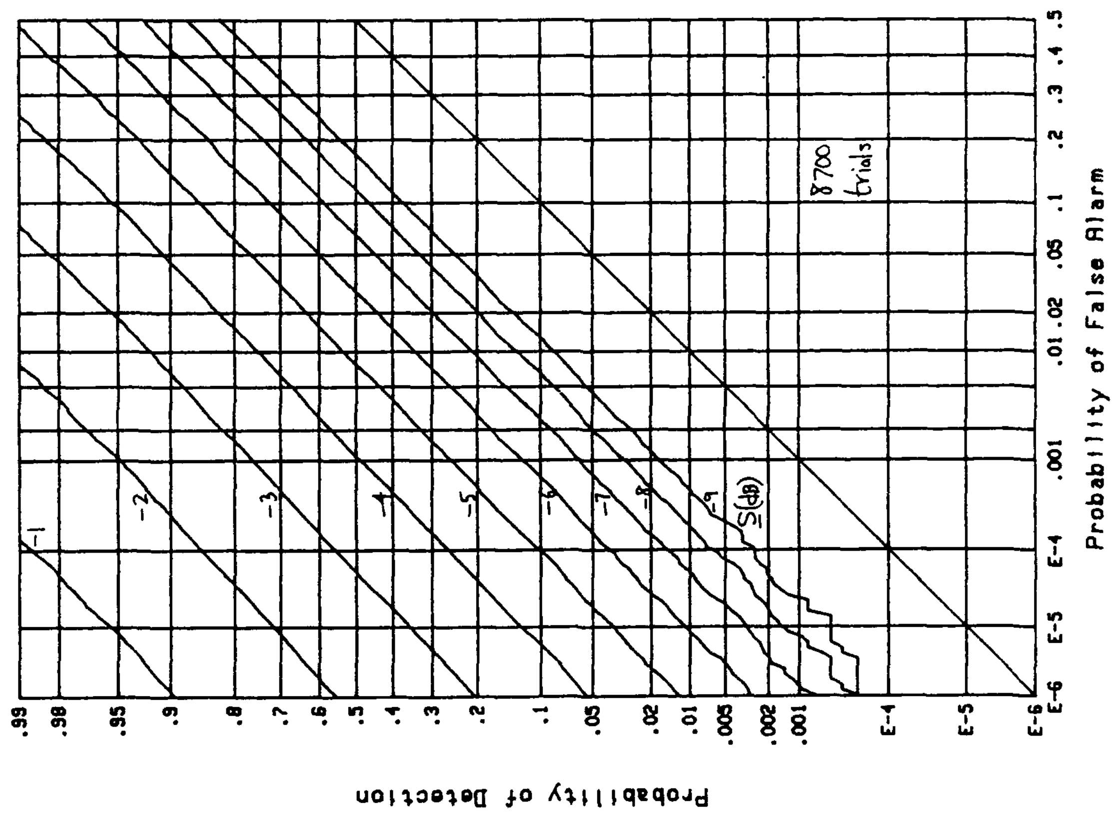

Figure E-99. ROC for SOML, M=256, M=256

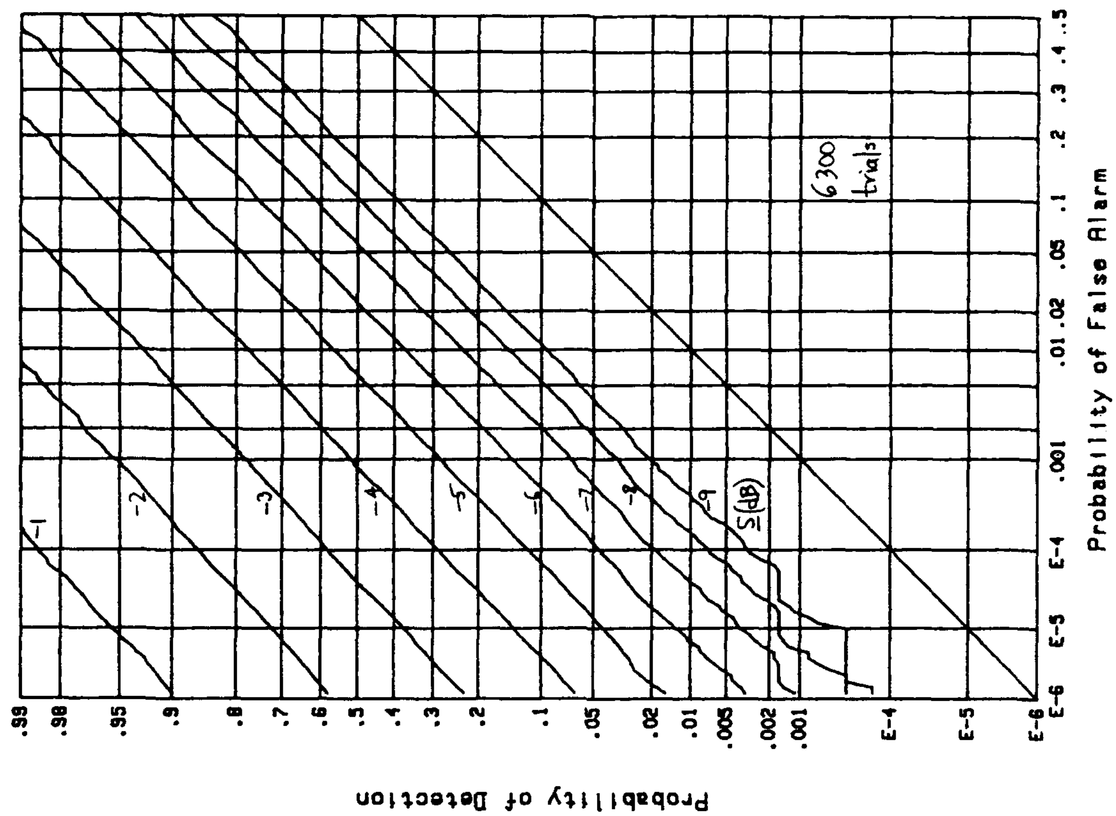

Figure E-100. ROC for SOML, M=256, M=512

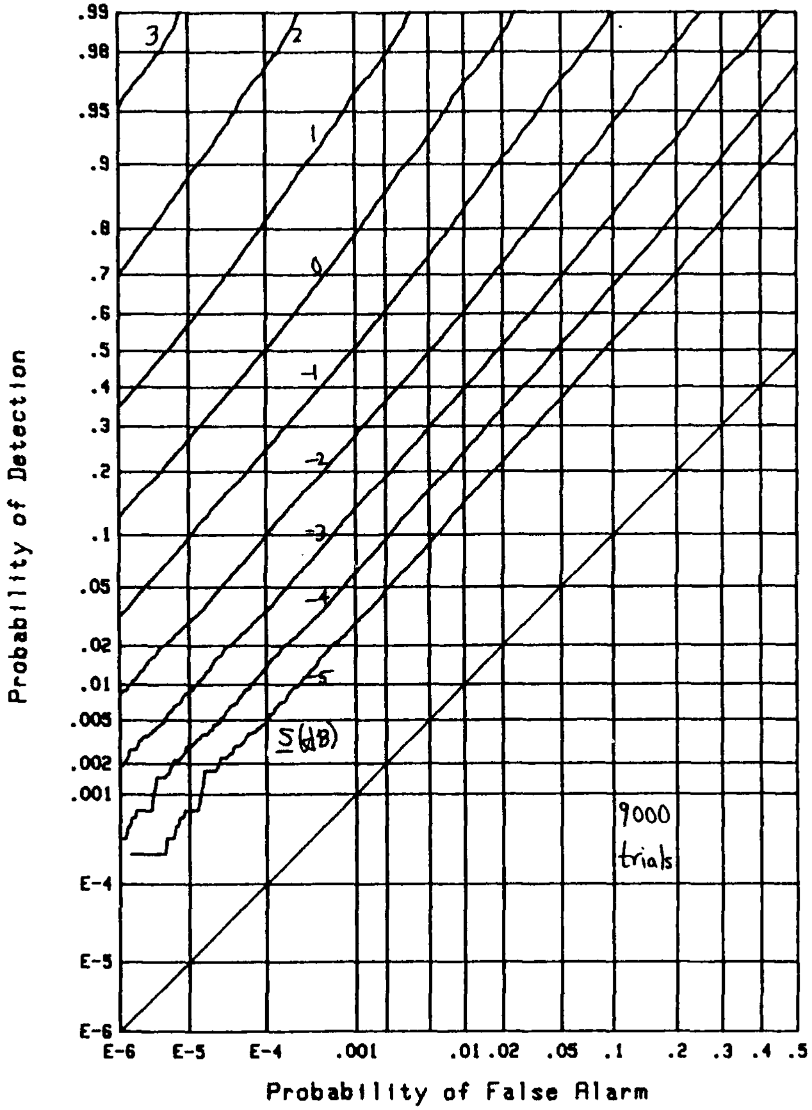

Figure E-101. ROC for SOML, M=512, M=2

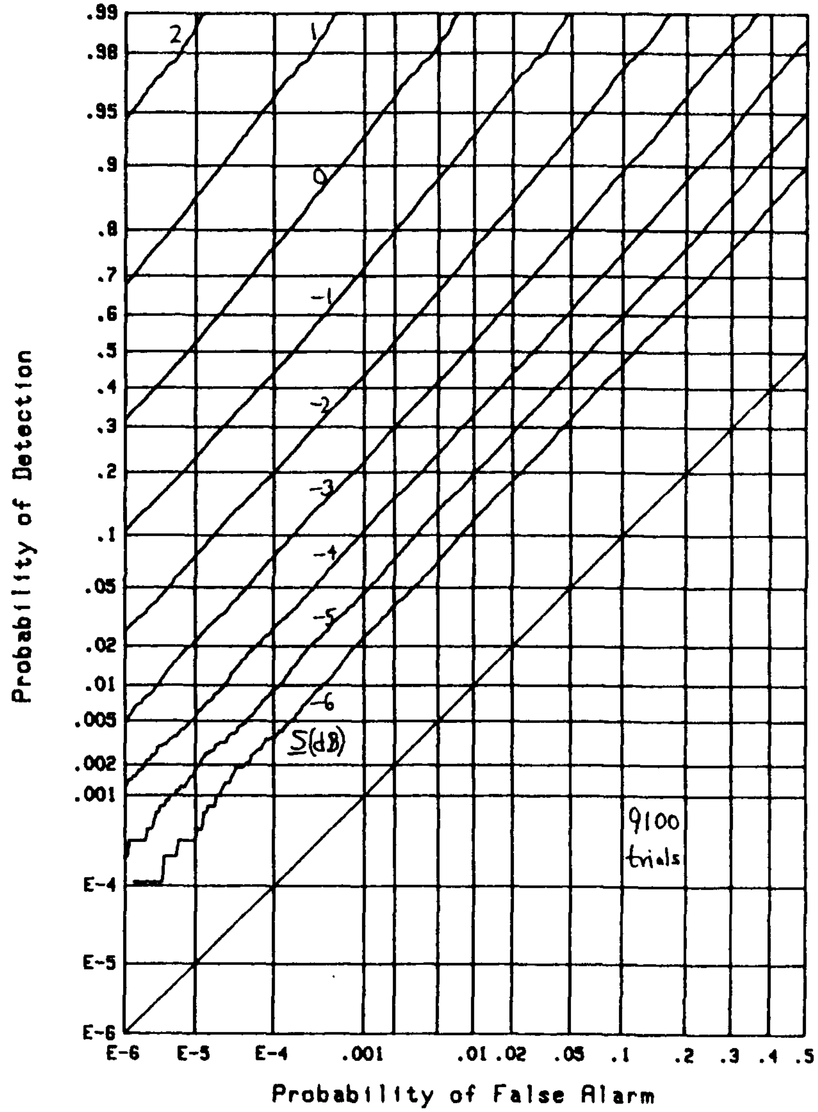

Figure E-102. ROC for SOML, M=512, M=3

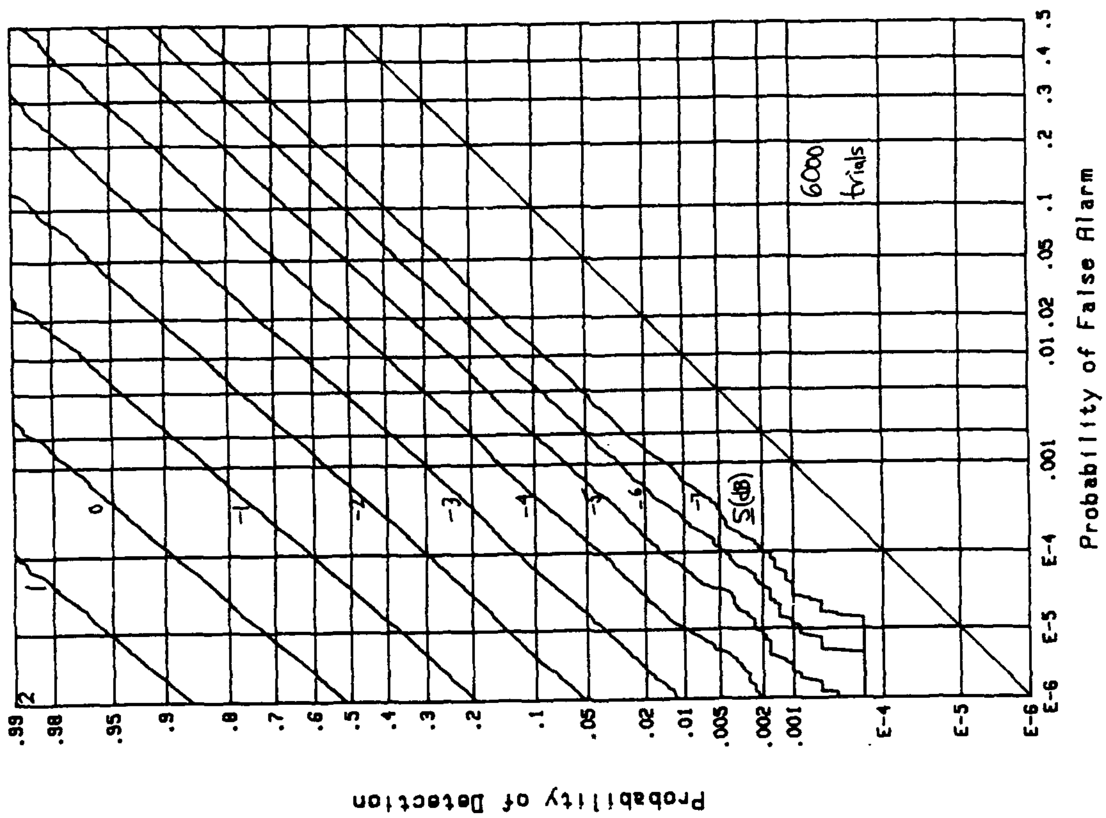

Figure E-103. ROC for SOML, M=512, M=4

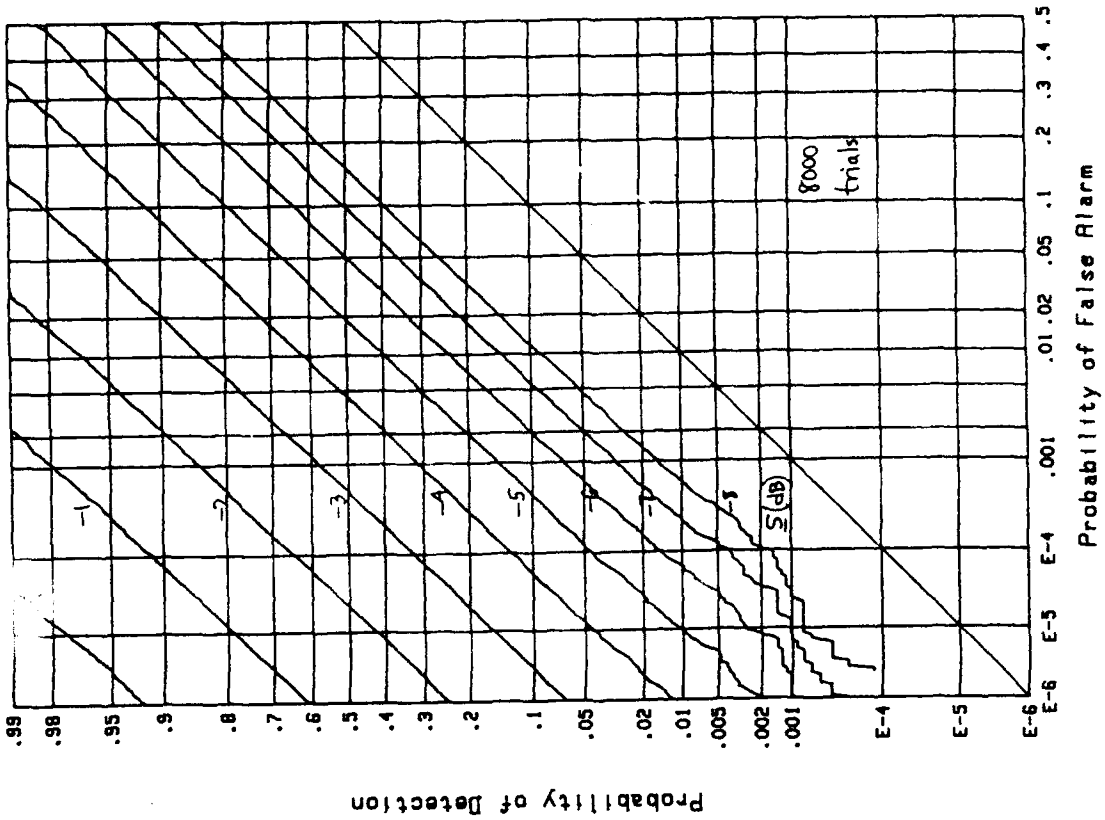

Figure E-104. ROC for SOML, M=512, M=8

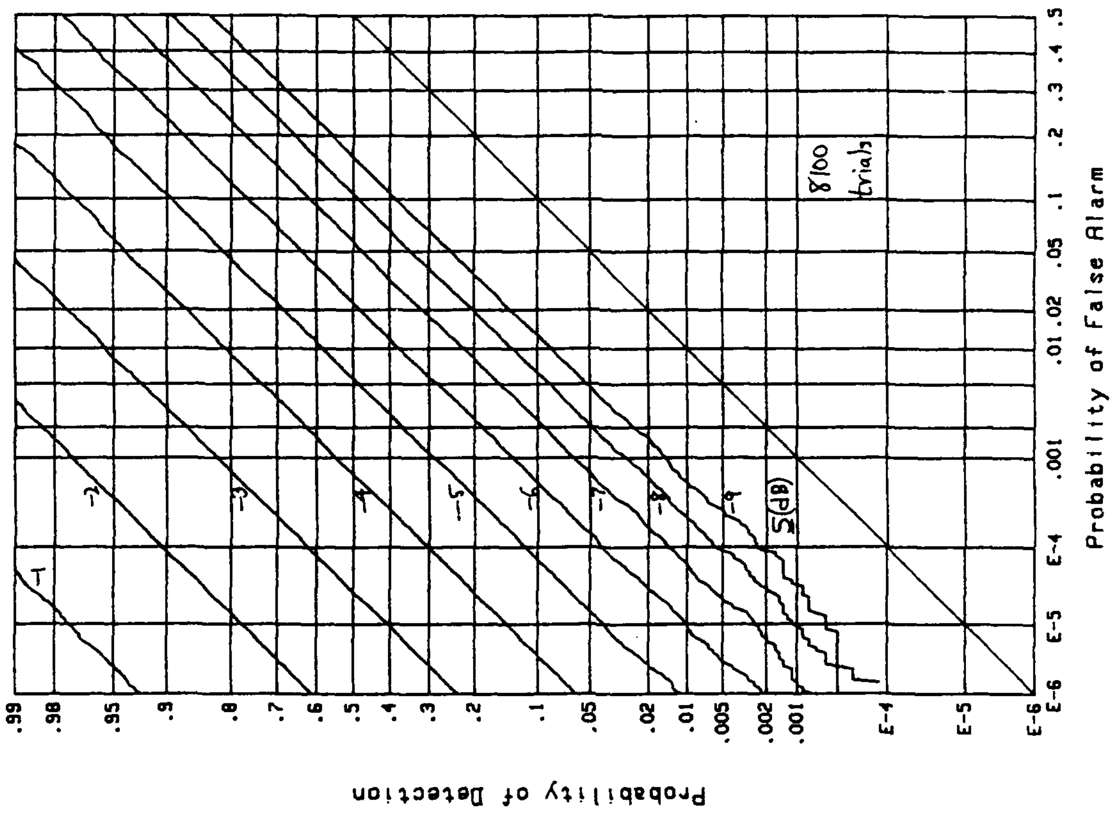

Figure E-105. ROC for SOML, $\underline{M}$=512, M=16

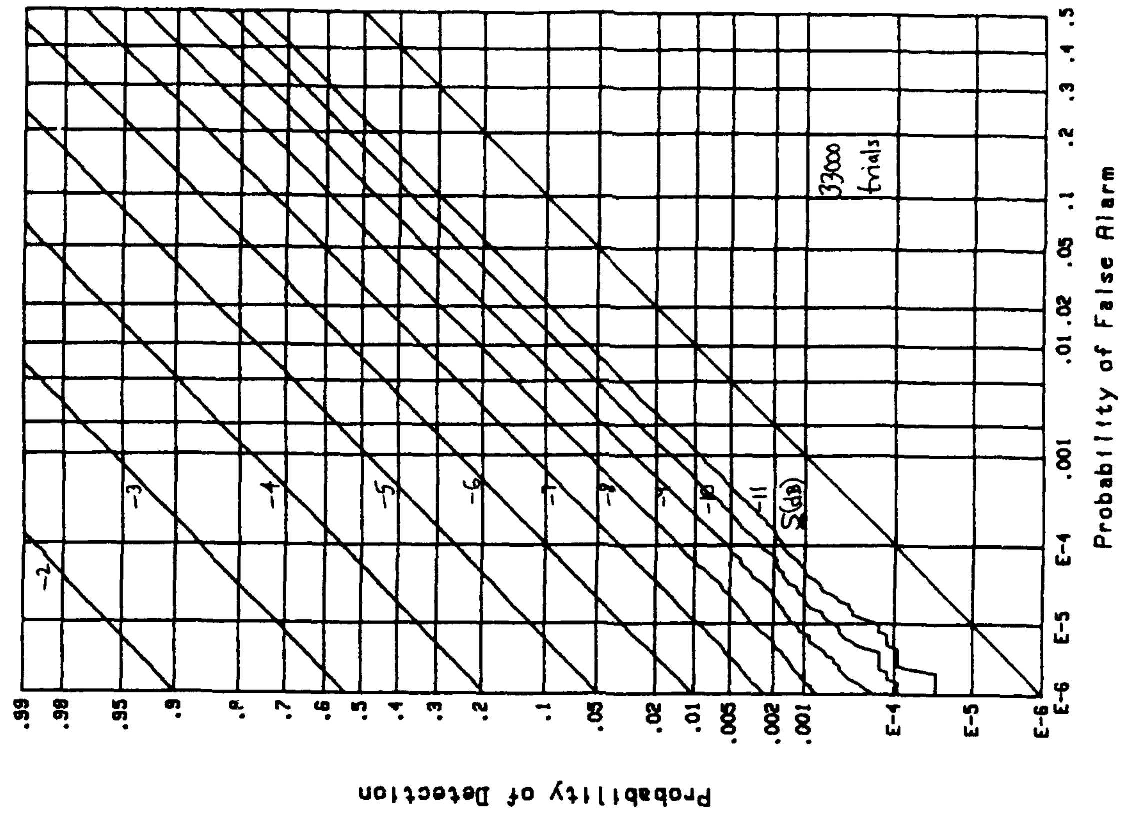

Figure E-106. ROC for SOML, $\underline{M}$=512, M=32

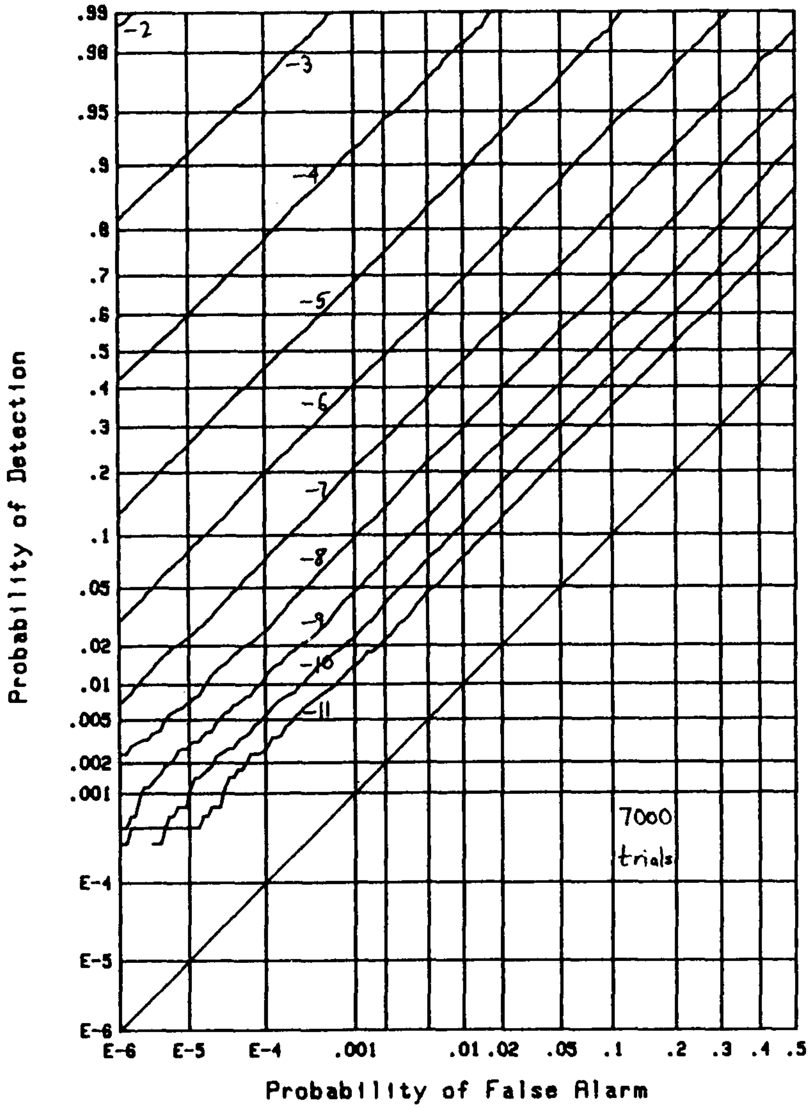

Figure E-107. ROC for SOML, M=512, M=64

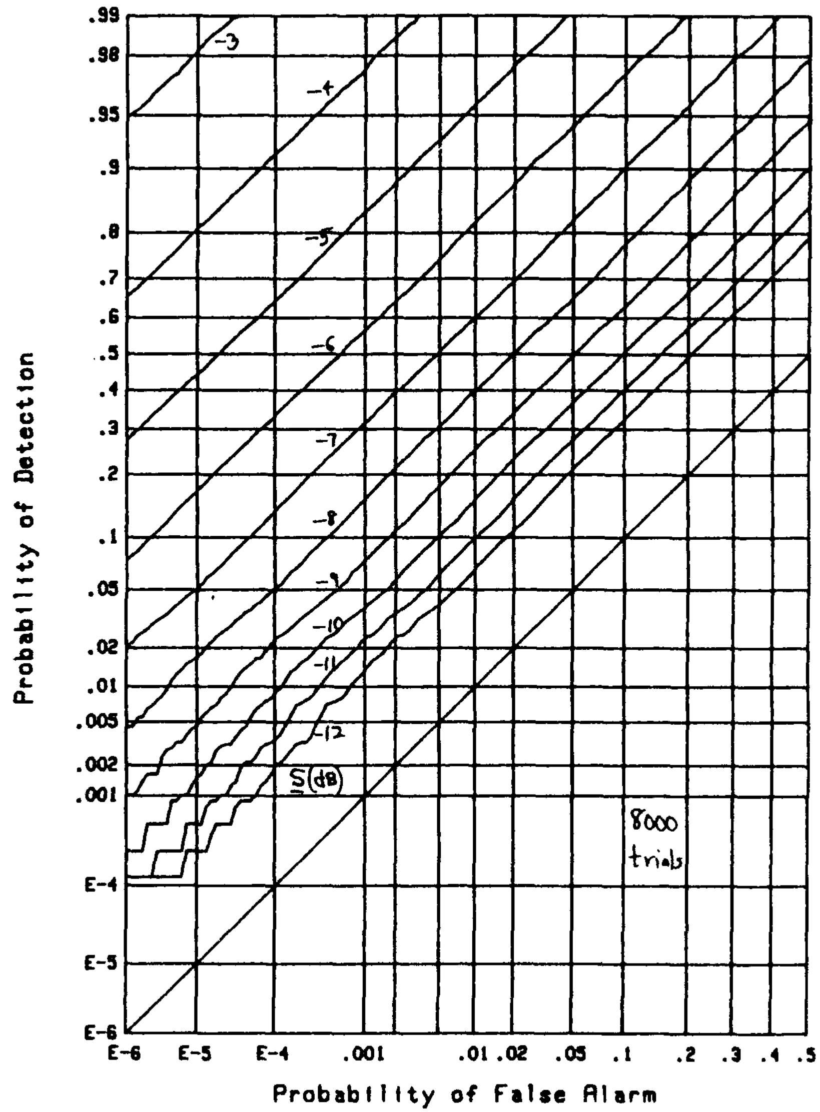

Figure E-108. ROC for SOML, M=512, M=128

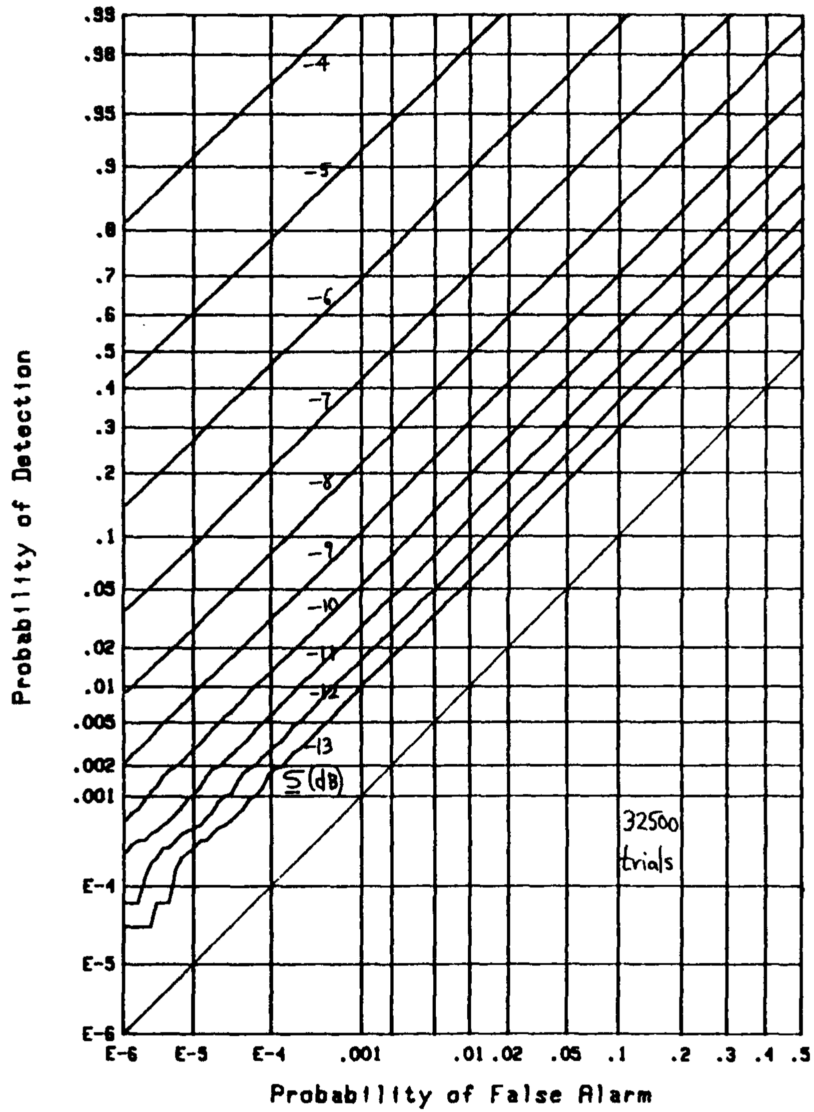

Figure E-109. ROC for SOML, $\underline{M}$=512, M=256

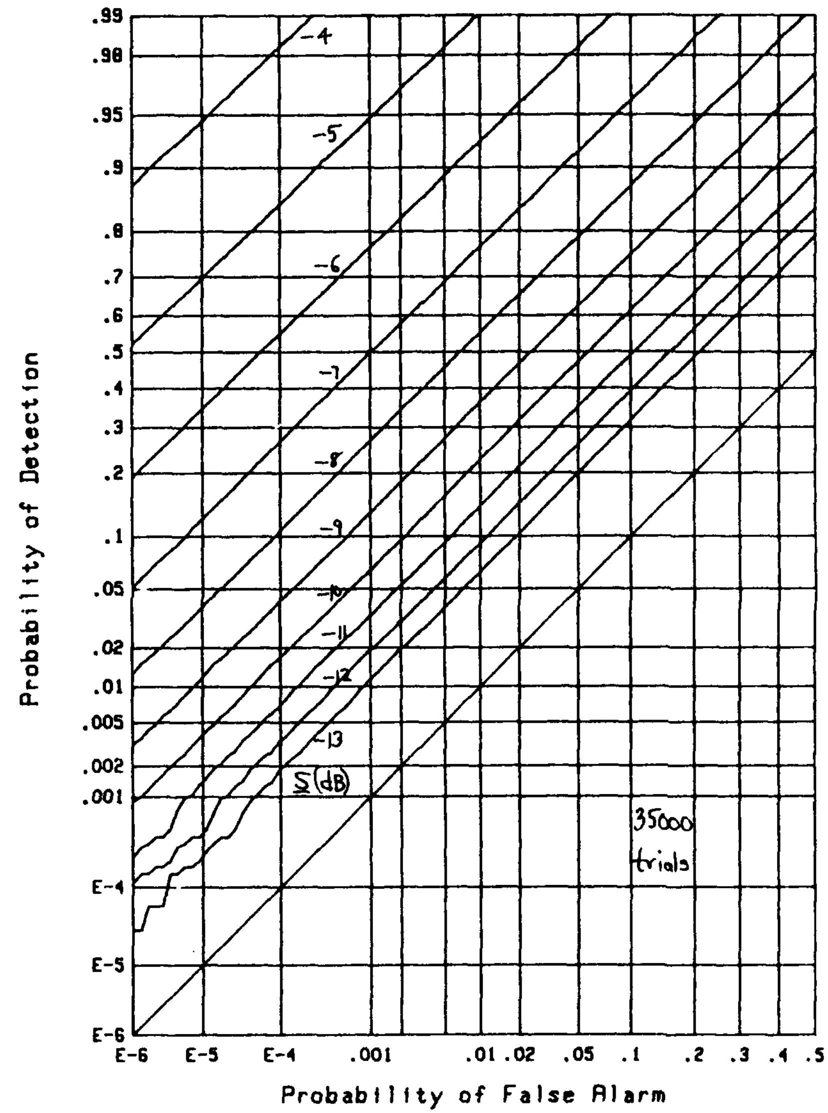

Figure E-110. ROC for SOML, $\underline{M}$=512, M=512

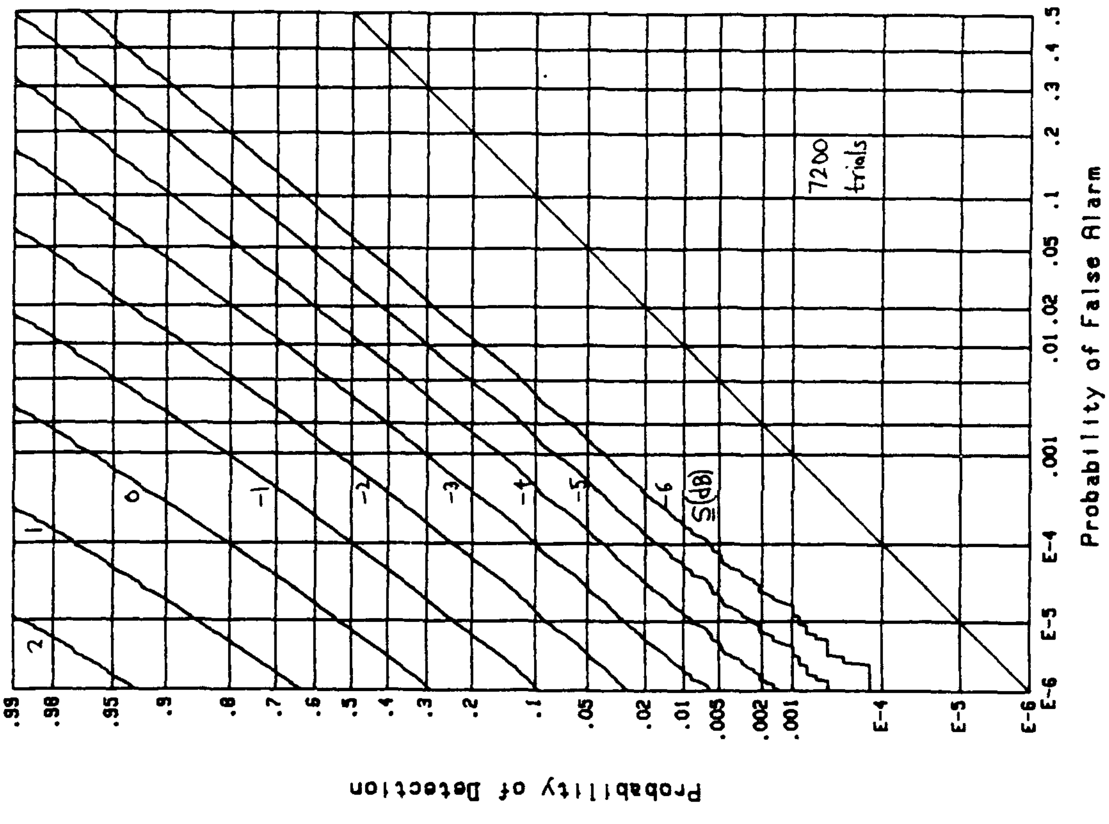

Figure E-111. ROC for SOML, $\underline{M}$=1024, M=2

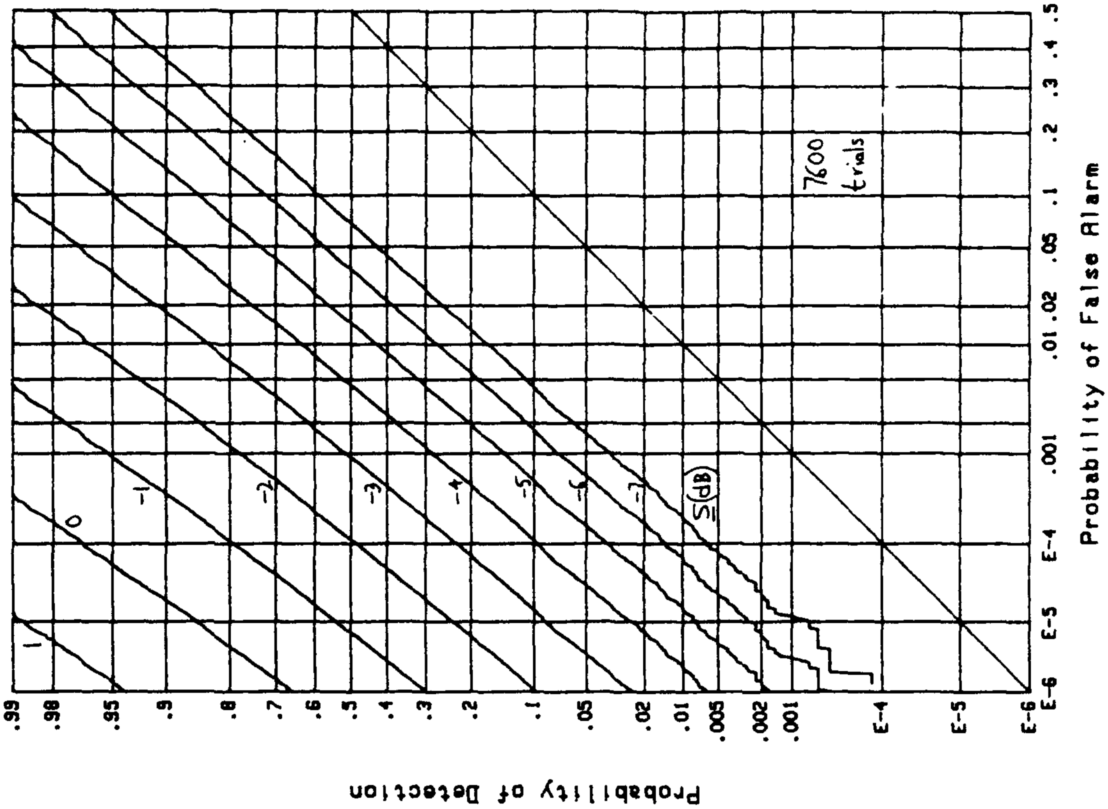

Figure E-112. ROC for SOML, $\underline{M}$=1024, M=3

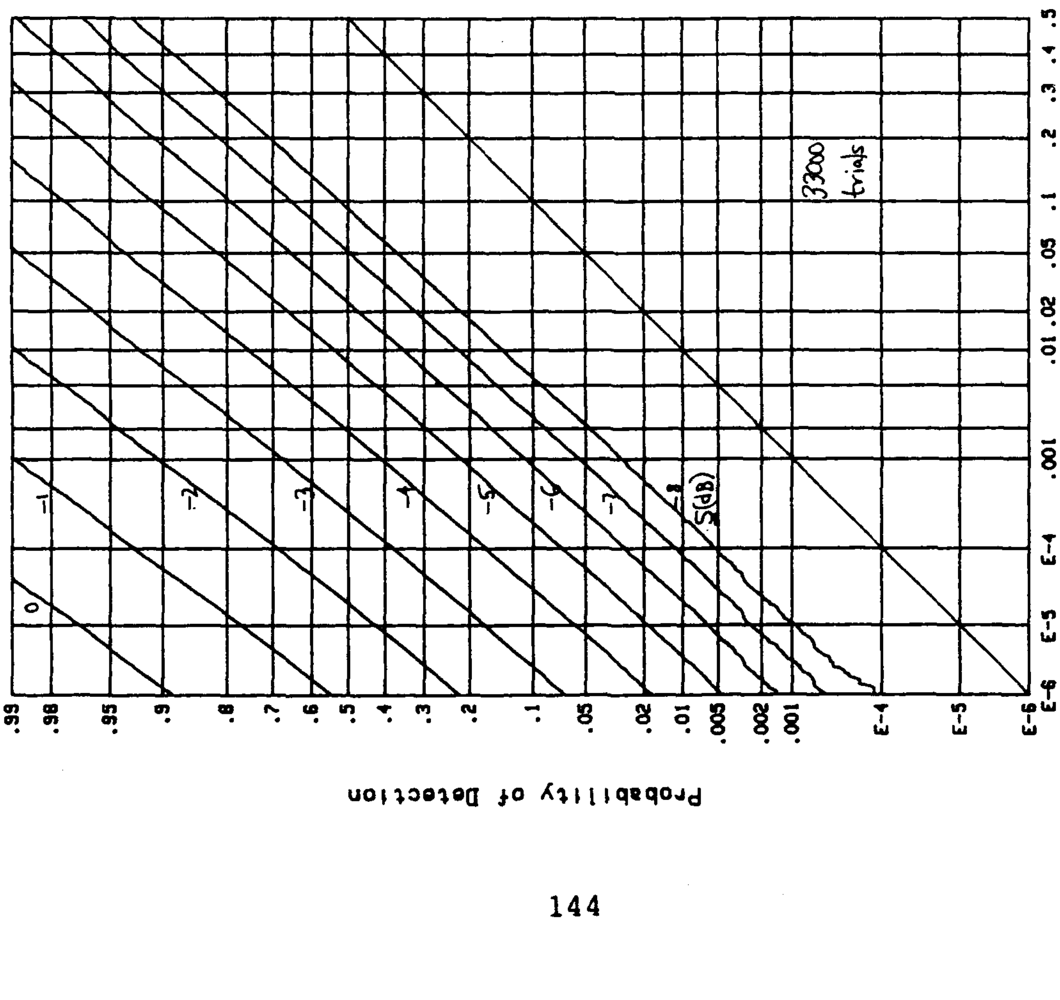

Figure E-113. ROC for SOML, M=1024, M=4

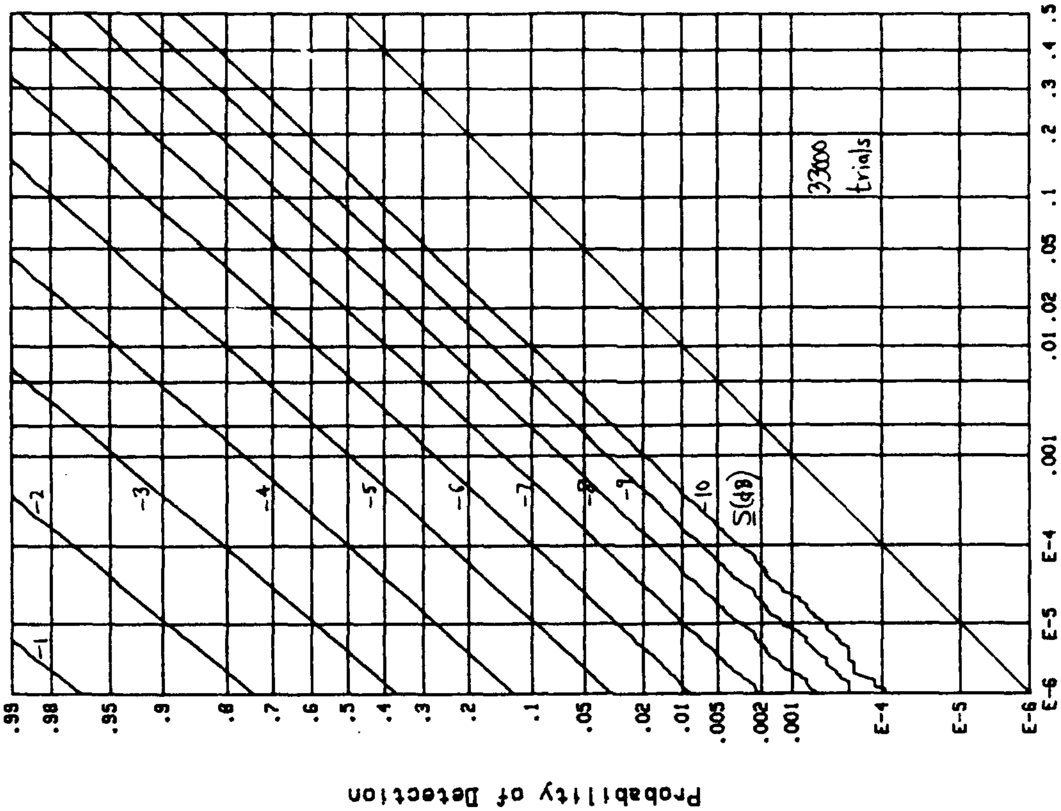

Figure E-114. ROC for SOML, M=1024, M=8

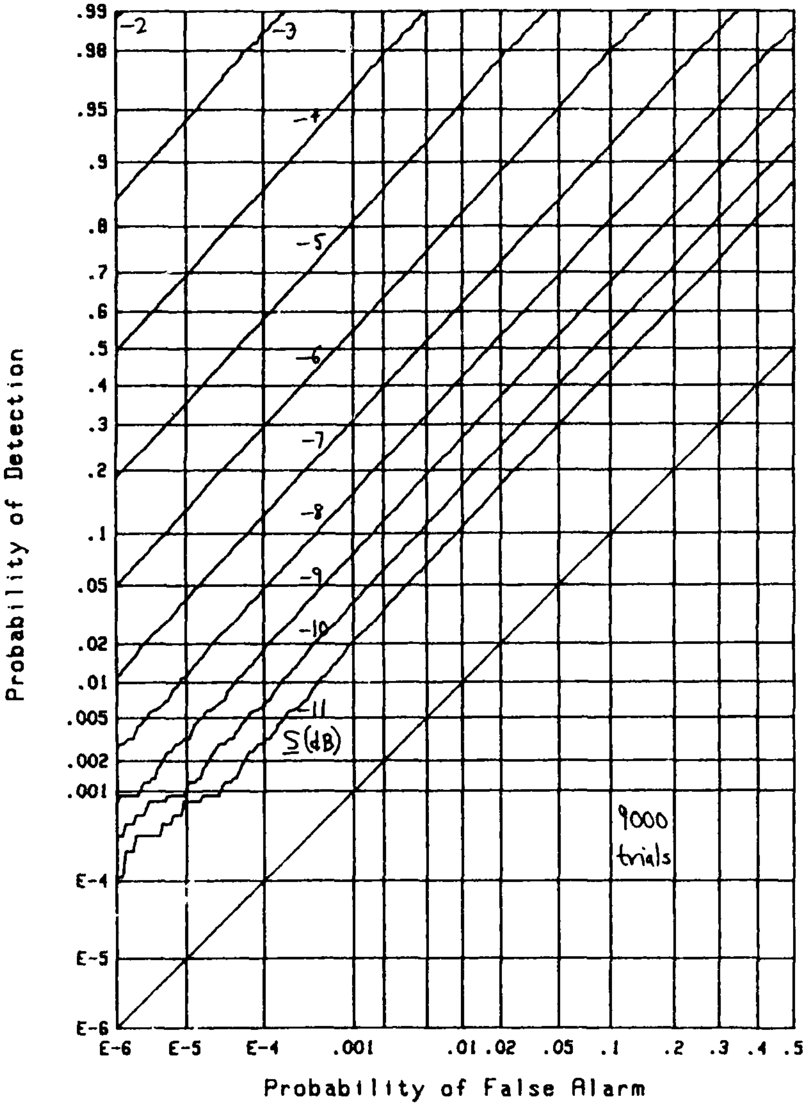

Figure E-115. ROC for SOML, M=1024, M=16

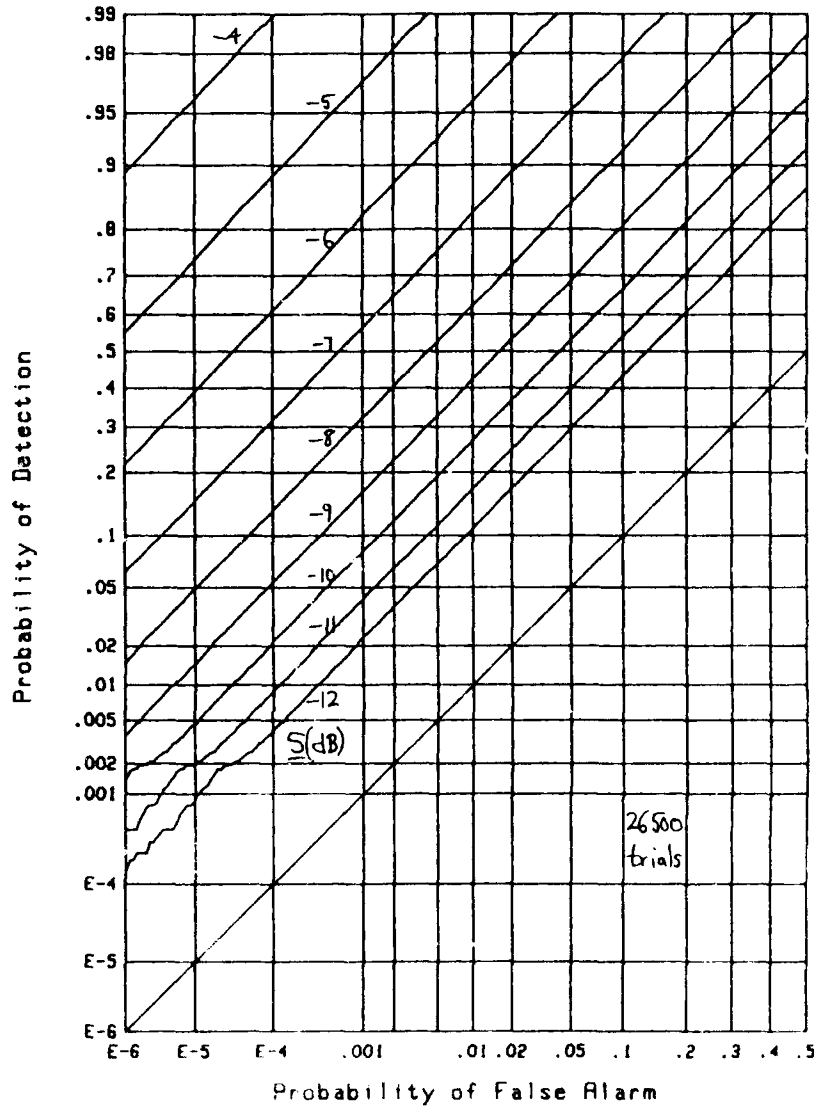

Figure E-116. ROC for SOML, M=1024, M=32

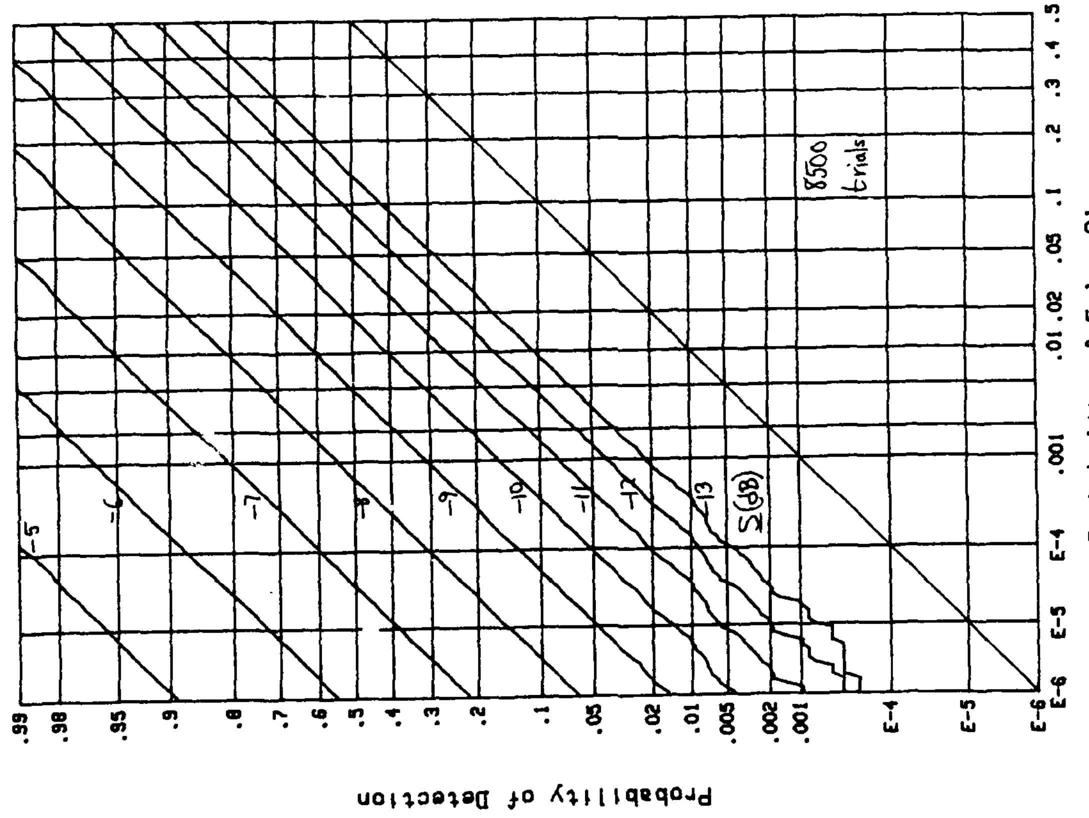

Figure E-117. ROC for SOML, M=1024, M=64

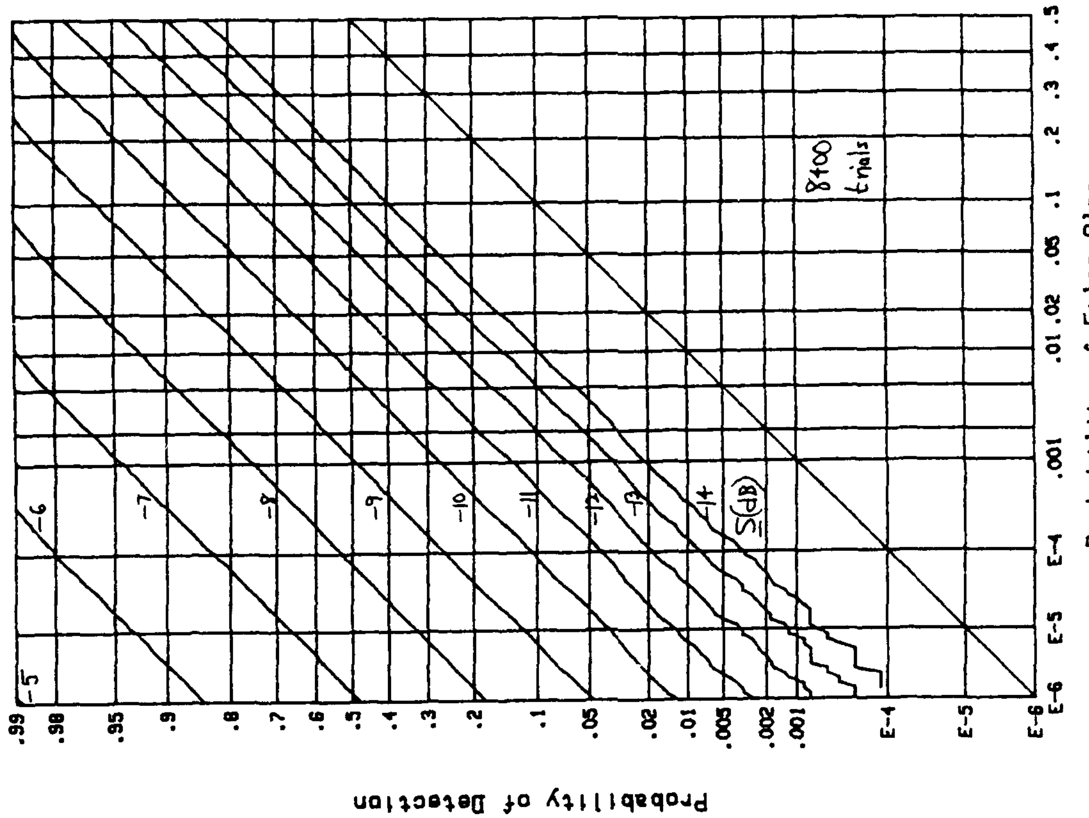

Figure E-118. ROC for SOML, M=1024, M=128

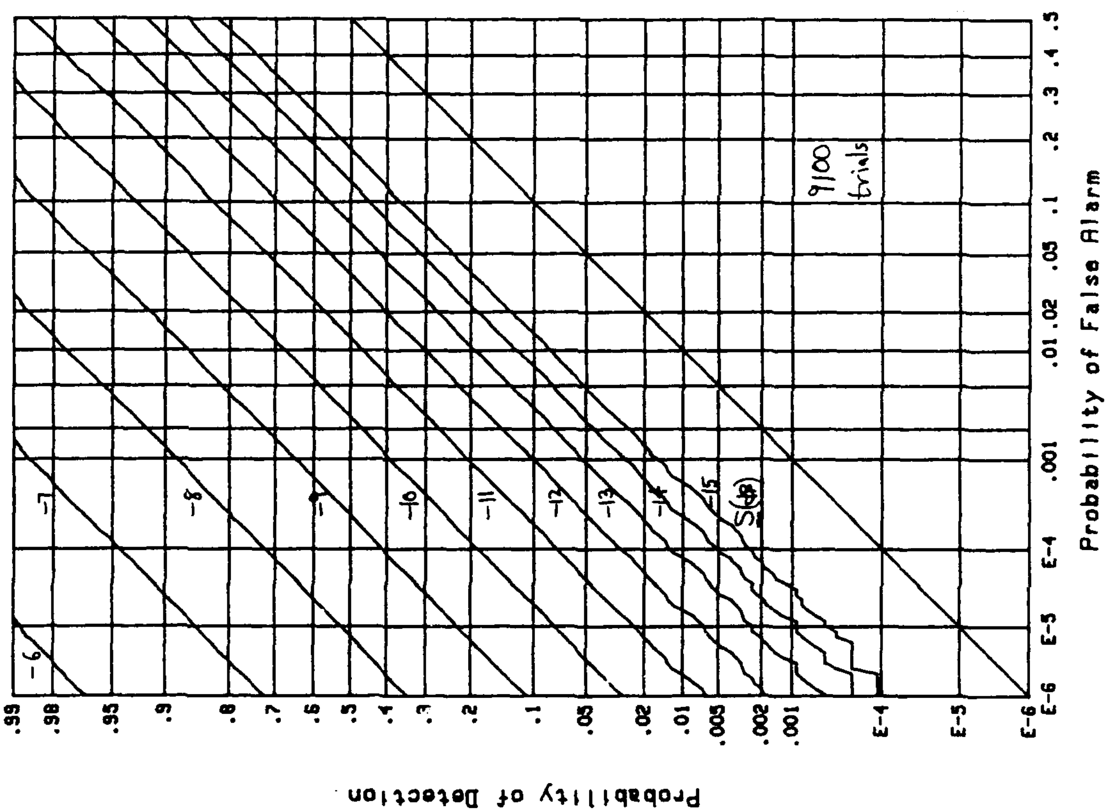

Figure E-119. ROC for SOML, M=1024, M=256

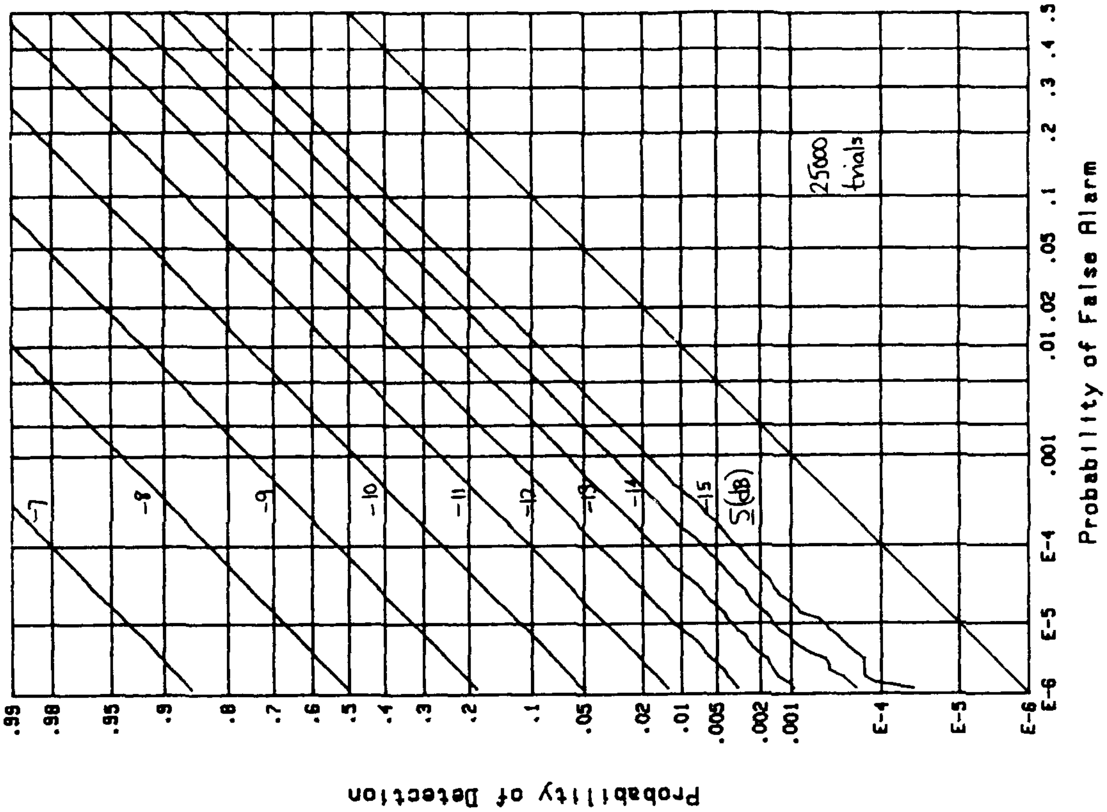

Figure E-120. ROC for SOML, M=1024, M=512

APPENDIX F. ON THE CHOICE OF M WHEN $\underline{M}$ IS KNOWN

The total number of search bins is N, and the number of occupied signal bins (when signal is present) is $\underline{M}$, which is <u>known</u>. However, the locations $\underline{L}$ of these $\underline{M}$ bins are totally unknown. Under both hypotheses H_0 and H_1, the average noise power in all N bins is 1; under H_1, the average signal power in the $\underline{M}$ occupied signal bins is common value $\underline{S}$, which is unknown.

The GLR test is to sum the $\underline{M}$ largest values of the input data $\{x_n\}$ and compare the sum with a threshold. This is equivalent to ordering the input data into the set $\{x'_n\}$, where $x'_n \geq x'_{n+1}$ for $1 \leq n \leq N-1$, summing the first $\underline{M}$ values, and comparing the sum with the same threshold. Although this test appears to agree with physical intuition, simulations reveal that better performance, in terms of P_d versus P_f, are obtained if the largest M bin outputs are summed, where M is sometimes different from $\underline{M}$. This means that the GLR test is not optimum in this situation; of course, the GLR procedure makes no claim for optimality, although it frequently leads to a high quality test. The explanation and remedy to this apparent discrepancy follows.

Under H_0, the average level of x_n is 1. On the other hand, under H_1, the average level of x_n is changed to $1+\underline{S}$ in the $\underline{M}$ occupied signal bins, but <u>only</u> for these $\underline{M}$ bins which contain signal. The remaining N-$\underline{M}$ bins still have average level 1 under H_1. Unfortunately, since the information about which particular bins are occupied is unknown, we are led to consider the ordered data $\{x'_n\}$, which contain identical information to $\{x_n\}$ under this situation. Even if we ignore any guidance from the GLR procedure, a justification for considering ordered data is that the presence of signal results in $\underline{M}$ larger data values on average, and therefore, larger values should get more weight.

For the ordered data $\{x'_n\}$, the situation regarding average values is somewhat different. Under H_0, the average level of x'_n decreases monotonically with n. When signal is added to $\underline{M}$ bins under H_1 and the data $\{x_n\}$ ordered, the average levels of <u>all</u> N data values x'_n are increased, not just the first $\underline{M}$ bins. In order to demonstrate this claim, let the first $\underline{M}$ bins contain the signal; this is no loss of generality, since we are not going to use this fact in our data processing. Now, under H_1, the act of ordering the measured data $\{x_n\}$ evicts some of the smaller signal members from their initial locations in set $[1,\underline{M}]$ into new quarters in the set of numbers $[\underline{M}+1,N]$ in data sequence $\{x'_n\}$. Thus, when H_1 is prevalent, the signal addition and bin movement raises the average levels of $\{x'_n\}$, not just for $1 \le n \le \underline{M}$, but also for some n values larger than $\underline{M}$. Some examples have shown increases all the way out to $n = N$.

For improved performance, we must be willing to look for average deflections in any and all bins. Therefore, even though we may know that only $\underline{M}$ bins are occupied by signal, we must process M bins of the <u>ordered</u> data $\{x'_n\}$, where M can be larger (or smaller) than $\underline{M}$, due to the spillover effect described above. In order to settle on near-optimum values of M to use, it is necessary to conduct simulations of the receiver operating characteristics for various combinations of N, $\underline{S}$, $\underline{M}$, and M.

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